MICROLOCALIZATION OF RATIONAL CHEREDNIK ALGEBRAS

MASAKI KASHIWARA AND RAPHAËL ROUQUIER

Abstract. We construct a microlocalization of the rational Cherednik algebras H of type S_n. This is achieved by a quantization of the Hilbert scheme $\mathrm{Hilb}^n\mathbb{C}^2$ of n points in $\mathbb{C}^2$. We then prove the equivalence of the category of H-modules and the one of modules over its microlocalization under certain conditions on the parameter.

Contents

1. Introduction

Let us recall that $\mathrm{Hilb}^n\mathbb{C}^2$, the Hilbert scheme of n points in $\mathbb{C}^2$, is a symplectic (in particular crepant) resolution of $\mathbb{C}^{2n}/S_n = S^n\mathbb{C}^2$. On the other hand, the orbifold $[\mathbb{C}^{2n}/S_n]$ (or the corresponding algebra $\mathbb{C}[\mathbb{C}^{2n}]\rtimes S_n$) is a non-commutative crepant resolution of $\mathbb{C}^{2n}/S_n$.

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There is an equivalence between derived categories of coherent sheaves on $\mathrm{Hilb}^n\mathbb{C}^2$ and finitely generated modules over $\mathbb{C}[\mathbb{C}^{2n}] \rtimes S_n$ (McKay's correspondence, cf. [11]).

The rational Cherednik algebra H_c associated with S_n is a one-parameter quantization of $\mathbb{C}[\mathbb{C}^{2n}] \rtimes S_n$. We construct a one-parameter quantization $\widetilde{\mathscr{A}_c}$ of $\mathscr{O}_{\mathrm{Hilb}^n\mathbb{C}^2}$ and an equivalence of categories between a certain category of $\widetilde{\mathscr{A}_c}$-modules (good modules with F-action) and the category of finitely generated H_c-modules (under certain conditions on the parameter c). Note that this is an equivalence of abelian categories, while the non-quantized McKay's correspondence is only an equivalence of derived categories.

The quantization $\widetilde{\mathscr{A}_c}$ is a sheaf over $\mathrm{Hilb}^n\mathbb{C}^2$. Locally on an open subset isomorphic to T^*U, it is isomorphic to the sheaf of micro-differential operators $\mathscr{W}$ with a homogenizing parameter $\hbar$.

Note that our construction is an analog of the Beilinson-Bernstein localization Theorem for universal enveloping algebras upon flag varieties:

nilpotent cone $\mathscr{N}$	$\mathbb{C}^{2n}/S_n$
enveloping algebra quotients $U_\lambda(\mathfrak{g})$	H_c
$T^*(G/B)$	$\mathrm{Hilb}^n\mathbb{C}^2$
$\mathscr{D}_{G/B.\lambda}$	$\widetilde{\mathscr{A}_c}$

$$
\begin{tikzcd}
\mathscr{D}_{G/B.\lambda} \arrow[r, leftrightarrow, dotted, "\sim"] \arrow[d, dotted, "\text{quantization}"'] & U_\lambda(\mathfrak{g}) \arrow[d, dotted, "\text{quantization}"] \\
T^*(G/B) \arrow[r, "\text{resolution}"'] & \mathscr{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathscr{A}_c} \arrow[r, leftrightarrow, dotted, "\sim"] \arrow[d, dotted, "\text{quantization}"'] & H_c \arrow[d, dotted, "\text{quantization}"] \\
\mathrm{Hilb}^n\mathbb{C}^2 \arrow[r, "\text{resolution}"'] & \mathbb{C}^{2n}/S_n
\end{tikzcd}
$$

Let us mention that our constructions give rise to the spherical subalgebra eH_ce of H_c and under certain assumptions on c the two algebras are Morita equivalent. It would be interesting to quantize directly the Procesi bundle to obtain H_c.

Let us now describe some earlier results related to our work. An important achievement of Etingof and Ginzburg [6] and of Gan and Ginzburg [7] is a construction of a deformation of the Harish-Chandra morphism for $\mathrm{GL}_n(\mathbb{C})$, providing a construction of the spherical subalgebra eH_ce of H_c as a quantum Hamiltonian reduction. This provides a quantization of the Calogero-Moser space, which is itself obtained by classical Hamiltonian reduction (Kazhdan, Kostant and Sternberg [21]).

Gordon and Stafford [8, 9] construct a one-parameter family of graded ($\mathbb{Z}$)-algebras $\mathcal{B}_c$ that quantize (a graded ($\mathbb{Z}$)-algebra Morita-equivalent to) the homogeneous coordinate ring of $\mathrm{Hilb}^n\mathbb{C}^2$.

In positive characteristic, Bezrukavnikov, Finkelberg and Ginzburg [4] construct a sheaf of Azumaya algebras on the Hilbert scheme whose algebra of global sections is isomorphic to H_c and obtain an equivalence of derived categories between modules over that Azumaya algebra and representations of H_c.

Let us explain the type of sheaf of algebras used to quantize $\mathrm{Hilb}^n\mathbb{C}^2$. On a complex contact manifold, Kashiwara [16] constructed the stack $\mathcal{E}$ of microdifferential operators. Locally, a model for a contact manifold is the projectivized cotangent bundle P^*X and

the stack $\mathcal{E}$ comes from the sheaf $\mathcal{E}_X$ of microdifferential operators of Sato, Kawai and Kashiwara.

On a symplectic variety, Kontsevich [22] and Polesello-Schapira [24] defined a stack $\mathscr{W}$ of microdifferential operators with a homogenizing parameter $\hbar$ (making all objects modules over $\mathbb{C}((\hbar))$). Locally, a model is T^*X and $\mathscr{W}$ comes from microdifferential operators on $P^*(X \times \mathbb{C})$ which do not depend on the extra variable.

For applications to representation theory, these constructions are unsatisfactory:

- the first construction "forgets about the zero-section"
- the second construction gives "too large" objects (defined over $\mathbb{C}((\hbar))$ instead of $\mathbb{C}$).

To overcome these difficulties, we consider here symplectic manifolds X with a $\mathbb{C}^\times$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight. We consider the case where the stack $\mathscr{W}$ comes from a sheaf of algebras together with a compatible action of $\mathbb{C}^\times$ and study the corresponding structure, a "W-algebra with F-action". The category of its modules is defined over $\mathbb{C}$, as the F-action induces a $\mathbb{C}^\times$- action on $\mathbb{C}((\hbar))$ whose invariant field is $\mathbb{C}$.

Let us now describe the structure of the paper.

In the first part of this paper (§ 2), we study a general setting for the quantization of symplectic manifolds X with a $\mathbb{C}^\times$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight. We first review the theory of W-algebras on symplectic manifolds (§ 2.2). In § 2.3, we introduce the notion of "W-algebra with F-action". An important point of this construction is that the category of $\mathscr{W}$-modules with F-action on a cotangent bundle (for the canonical structure) is equivalent to the category of modules over the sheaf $\mathscr{D}$ of differential operators. We adapt in § 2.4 the study of equivariance and its twisted version for the action of a complex Lie group and we explain how to construct W-algebras with F-action by symplectic reduction in § 2.5. Finally, in § 2.6 we provide sufficient conditions to ensure $\mathscr{W}$-affinity (a counterpart of Beilinson-Bernstein's result for $\mathscr{D}$-modules).

Section 3 is devoted to the construction of $\mathscr{D}$-modules with an action of the rational Cherednik algebra H_c of type A_{n-1} or of its spherical subalgebra eH_ce. This is related to the constructions of [4, 7]. Let $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. We construct (§ 3.2) a quasi-coherent $\mathscr{D}_{\mathfrak{g}\times V}$-module $\mathscr{M}_c$ together with an action of H_c, building on the explicit description of the $\mathscr{D}$-module arising in Springer's correspondence given in [14]. We construct a coherent $\mathscr{D}_{\mathfrak{g}\times V}$-submodule $\mathscr{L}_c$ of $\mathscr{M}_c$ that is stable under the action of the spherical subalgebra of H_c and we construct a shift operator (§ 3.3). This is achieved by reduction to rank 2.

In § 4 we construct a W-algebra with F-action on $\mathrm{Hilb}^n\mathbb{C}^2$ by symplectic reduction from the previous constructions. After recalling some properties of $\mathrm{Hilb}^n\mathbb{C}^2$ in § 4.1, we construct in § 4.2 a W-algebra $\widetilde{\mathscr{A}_c}$ on $\mathrm{Hilb}^n\mathbb{C}^2$ by symplectic reduction of $\mathscr{L}_c$ for the action of $\mathrm{GL}_n(\mathbb{C})$. In § 4.3, we present our main results: $\widetilde{\mathscr{A}_c}$-affinity of $\mathrm{Hilb}^n\mathbb{C}^2$, an isomorphism between global sections of $\widetilde{\mathscr{A}_c}$ and the spherical algebra and an equivalence between the category of good $\widetilde{\mathscr{A}_c}$-modules with F-action and the one of finitely generated modules over the spherical algebra. We also describe similar results for H_c. So, we have obtained a microlocalization of the rational Cherednik algebras: we have constructed a W-algebra with F-action over the Hilbert scheme whose algebra of global sections is isomorphic

to H_c and whose modules are equivalent to representations of H_c. Those results are obtained under certain assumptions on c. We explain in § 4.4 how to view sections of our W-algebras over open subsets of the Hilbert schemes as appropriate fractions in the Cherednik algebra. Finally, we describe explicitly the constructions for $n = 2$ in § 4.5.

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2. F-actions on W-algebras

2.1. **Notations.** By a manifold M, we mean a complex manifold, equipped with the classical topology and $\mathscr{O}_M$ is the sheaf of holomorphic functions. We denote by $\mathscr{D}_M$ the sheaf of differential operators with holomorphic coefficients and by $\mathcal{E}_M$ the sheaf of formal micro-differential operators on the cotangent bundle T^*M.

We denote by $\mathbb{G}_{\mathrm{m}}$ the multiplicative group $\mathbb{C}^\times$.

Given a ring A, we denote by $\mathrm{Mod}_{\mathrm{coh}}(A)$ the category of coherent left A-modules.

2.2. **W-algebras.** We shall review some results on W-algebras. We refer the reader to [24] (where the convergent version is studied, while we use the simpler formal version).

2.2.1. Let $\mathbf{k} = \mathbb{C}((\hbar))$ be the field of formal Laurent series in an indeterminate $\hbar$ and let $\mathbf{k}(0) = \mathbb{C}[[\hbar]]$. Given $m \in \mathbb{Z}$, we define $\mathscr{W}_{T^*\mathbb{C}^n}(m)$ as the sheaf of formal series $\sum_{k \geqslant -m} \hbar^k a_k$ ($a_k \in \mathscr{O}_{T^*\mathbb{C}^n}$) on the cotangent bundle $T^*\mathbb{C}^n$ of $\mathbb{C}^n$ and we set $\mathscr{W}_{T^*\mathbb{C}^n} = \cup_m \mathscr{W}_{T^*\mathbb{C}^n}(m)$. Then, $\mathscr{W}_{T^*\mathbb{C}^n}$ has a structure of $\mathbf{k}$-algebra given by

$$a \circ b = \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \hbar^{|\alpha|} \frac{1}{\alpha!} \partial^\alpha_\xi a \cdot \partial^\alpha_x b.$$

We have a ring homomorphism $\mathscr{D}_{\mathbb{C}^n}(\mathbb{C}^n) \to \mathscr{W}_{T^*\mathbb{C}^n}(T^*\mathbb{C}^n)$ given by $x_i \mapsto x_i$, $\dfrac{\partial}{\partial x_i} \mapsto \hbar^{-1}\xi_i$.

2.2.2. Let X be a complex symplectic manifold with symplectic form ω_X. We denote by X^{opp} the symplectic manifold X with symplectic form $-\omega_X$.

A W-*algebra* is a $\mathbf{k}$-algebra $\mathscr{W}$ on X such that for any point $x \in X$, there are an open neighbourhood U of x, a symplectic map $f\colon U \to T^*\mathbb{C}^n$ and a $\mathbf{k}$-algebra isomorphism $g\colon \mathscr{W}|_U \xrightarrow{\sim} f^{-1}\mathscr{W}_{T^*\mathbb{C}^n}$.

A W-algebra $\mathscr{W}$ satisfies the following properties.

(i) The algebra $\mathscr{W}$ is a coherent and noetherian algebra.
(ii) $\mathscr{W}$ contains a canonical subalgebra $\mathscr{W}(0)$ which is locally isomorphic to $\mathscr{W}_{T^*\mathbb{C}^n}(0)$ (via the maps g). We set $\mathscr{W}(m) = \hbar^{-m}\mathscr{W}(0)$.
(iii) We have a canonical $\mathbb{C}$-algebra isomorphism $\mathscr{W}(0)/\mathscr{W}(-1) \xrightarrow{\sim} \mathscr{O}_X$ (coming from the canonical isomorphism via the maps g). The corresponding morphism $\sigma_m\colon \mathscr{W}(m) \to \hbar^{-m}\mathscr{O}_X$ is called the *symbol map*.
(iv) We have
$$\sigma_0(\hbar^{-1}[a, b]) = \{\sigma_0(a), \sigma_0(b)\}$$
for any a, $b \in \mathscr{W}(0)$. Here $\{\bullet, \bullet\}$ is the Poisson bracket.
(v) The canonical map $\mathscr{W}(0) \to \varprojlim_{m\to\infty} \mathscr{W}(0)/\mathscr{W}(-m)$ is an isomorphism.
(vi) A section a of $\mathscr{W}(0)$ is invertible in $\mathscr{W}(0)$ if and only if $\sigma_0(a)$ is invertible in $\mathscr{O}_X$.

(vii) Given ϕ a $\mathbf{k}$-algebra automorphism of $\mathscr{W}$, we can find locally an invertible section a of $\mathscr{W}(0)$ such that $\phi = \mathrm{Ad}(a)$. Moreover a is unique up to a scalar multiple. In other words, we have canonical isomorphisms

$$\begin{tikzcd} \mathscr{W}(0)^\times/\mathbf{k}(0)^\times \arrow[r, "\sim", "\mathrm{Ad}"'] \arrow[d, "\sim"'] & \mathrm{Aut}(\mathscr{W}(0)) \arrow[d, "\sim"] \\ \mathscr{W}^\times/\mathbf{k}^\times \arrow[r, "\sim", "\mathrm{Ad}"'] & \mathrm{Aut}(\mathscr{W}). \end{tikzcd}$$

(viii) Let v be a $\mathbf{k}$-linear filtration-preserving derivation of $\mathscr{W}$. Then there exists locally a section a of $\mathscr{W}(1)$ such that $v = \mathrm{ad}(a)$. Moreover a is unique up to a scalar. In other words, we have an isomorphism

$$\mathscr{W}(1)/\hbar^{-1}\mathbf{k}(0) \xrightarrow[\mathrm{ad}]{\sim} \mathrm{Der}_{\mathrm{filtered}}(\mathscr{W}).$$

(ix) If $\mathscr{W}$ is a W-algebra, then its opposite ring $\mathscr{W}^{\mathrm{opp}}$ is a W-algebra on X^{opp}.

Conjecturally, (iii), (iv) and (v) characterize $\mathscr{W}(0)$.

Note that two W-algebras on X are locally isomorphic.

2.2.3. Assume there exist a_i, $b_i \in \mathscr{W}(0)$ $(i = 1, \dots, n)$ such that $[a_i, a_j] = [b_i, b_j] = 0$ and $[b_i, a_j] = \hbar\delta_{ij}$. They induce a symplectic map

$$f = (\sigma_0(a_1), \dots, \sigma_0(a_n); \sigma_0(b_1), \dots, \sigma_0(b_n)) : X \to T^*\mathbb{C}^n.$$

Then, there exists a unique isomorphism

$$\mathscr{W} \xrightarrow{\sim} f^{-1}\mathscr{W}_{T^*\mathbb{C}^n},\ a_i \mapsto x_i,\ b_i \mapsto \xi_i.$$

We call $(a_1, \dots, a_n; b_1, \dots, b_n)$ *quantized symplectic coordinates* of $\mathscr{W}$.

Let M be a complex manifold M and $\pi_M \colon T^*M \to M$ the projection. We can associate canonically a W-algebra $\mathscr{W}_{T^*M}$ with a morphism $\pi_M^{-1}\mathscr{D}_M \to \mathscr{W}_{T^*M}$ such that

$$\begin{tikzcd} \pi_M^{-1}F_m(\mathscr{D}_M) \arrow[r] \arrow[d, "\sigma_m"'] & \mathscr{W}_{T^*M}(m) \arrow[d, "\sigma_m"] \\ \mathscr{O}_{T^*M} \arrow[r, "\hbar^{-m}"'] & \hbar^{-m}\mathscr{O}_{T^*M} \end{tikzcd}$$

commutes. Here, $F(\mathscr{D}_M)$ is the order filtration of $\mathscr{D}_M$. Note that $\pi_M^{-1}\mathscr{D}_M \to \mathscr{W}_{T^*M}$ decomposes into $\pi_M^{-1}\mathscr{D}_M \to \mathcal{E}_M \to \mathscr{W}_{T^*M}$. The ring $\mathscr{W}_{T^*M}$ is flat over $\pi_M^{-1}\mathscr{D}_M$ and faithfully flat over $\mathcal{E}_M$. In particular, for a coherent $\mathscr{D}_M$-module $\mathscr{M}$, the characteristic variety $\mathrm{Ch}(\mathscr{M})$ coincides with $\mathrm{Supp}(\mathscr{W}_{T^*M} \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M})$.

Let X and Y be two symplectic manifolds. The product $X \times Y$ is also a symplectic manifold. For a W-algebra $\mathscr{W}_X$ on X and a W-algebra $\mathscr{W}_Y$ on Y, there is a W-algebra $\mathscr{W}_X \boxtimes \mathscr{W}_Y$ on $X \times Y$. Letting $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the projections, $\mathscr{W}_X \boxtimes \mathscr{W}_Y$ contains $p_1^{-1}\mathscr{W}_X \otimes_{\mathbf{k}} p_2^{-1}\mathscr{W}_Y$ as a $\mathbf{k}$-subalgebra, and is faithfully flat over it.

For a $\mathscr{W}$-module $\mathscr{M}$, a $\mathscr{W}(0)$-*lattice* is a coherent $\mathscr{W}(0)$-submodule $\mathscr{N}$ of $\mathscr{M}$ such that the canonical map $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{N} \to \mathscr{M}$ is an isomorphism.

We say that a $\mathscr{W}$-module $\mathscr{M}$ is *good* if for any relatively compact open subset U of X, there exists a coherent $\mathscr{W}(0)|_U$-lattice of $\mathscr{M}|_U$. The full subcategory of good $\mathscr{W}$-modules is an abelian subcategory of the category of $\mathscr{W}$-modules.

The following fact will be used in this paper (see [20, Theorem 1.2.2], where the convergent version is proved).

Lemma 2.1. *Let r be an integer and let $\mathscr{M}$ be a coherent $\mathscr{W}$-module such that $\mathscr{E}xt^j_{\mathscr{W}}(\mathscr{M},\mathscr{W})=0$ for any $j>r$. Then $\mathscr{H}^j_S(\mathscr{M})=0$ for any closed analytic subset S and any $j<\operatorname{codim} S-r$.*

Let $\bar{\mathbf{k}}:=\cup_{n>0}\mathbb{C}((\hbar^{1/n}))$ be an algebraic closure of $\mathbf{k}$. We will sometimes need to replace $\mathscr{W}$ with $\mathbf{k}'\otimes_{\mathbf{k}}\mathscr{W}$ for some field $\mathbf{k}'$ with $\mathbf{k}\subset\mathbf{k}'\subset\bar{\mathbf{k}}$.

2.3. **F-actions.**

2.3.1. Let X be a symplectic manifold. Consider an action of $\mathbb{G}_{\mathrm{m}}$ on X, viewed as a manifold: $\mathbb{C}^\times\ni t\mapsto T_t\in\operatorname{Aut}(X)$. We assume $\mathbb{G}_{\mathrm{m}}$ stabilizes the line $\mathbb{C}\omega_X\subset H^0(X,\Omega^2_X)$ with a positive weight m, i.e., $T_t^*\omega_X=t^m\omega_X$ for all $t\in\mathbb{C}^\times$.

We denote by v the vector field given by the $\mathbb{G}_{\mathrm{m}}$-action: $v(a)(x)=\frac{d}{dt}a(T_t(x))|_{t=1}$. The Poisson bracket $\{\bullet,\bullet\}$ is homogeneous of degree $-m$:

$$T_t^*\{a,b\}=t^{-m}\{T_t^*a,T_t^*b\}\text{ and }v\{a,b\}=\{v(a),b\}+\{a,v(b)\}-m\{a,b\}\text{ for }a,b\in\mathscr{O}_X.$$

Let $\mathscr{W}$ be a W-algebra.

Definition 2.2. *An* F-action with exponent m *on* $\mathscr{W}$ *is an action of* $\mathbb{G}_{\mathrm{m}}$ *on the* $\mathbb{C}$-*algebra* $\mathscr{W}$, $\mathcal{F}_t\colon T_t^{-1}\mathscr{W}\xrightarrow{\sim}\mathscr{W}$ *for* $t\in\mathbb{C}^\times$, *such that* $\mathcal{F}_t(\hbar)=t^m\hbar$ *and* $\mathcal{F}_t(a)$ *depends holomorphically on* t *for any* $a\in\mathscr{W}$.

Let us fix an F-action with exponent m on $\mathscr{W}$. The $\mathbb{G}_{\mathrm{m}}$-action induces an order-preserving derivation v_F of $\mathscr{W}$ given by $v_F(a)=\frac{d}{dt}\mathcal{F}_t(a)|_{t=1}$. It satisfies the following properties:

$$\begin{aligned}&v_F(\hbar)=m\hbar,\\&\sigma_0(v_F(a))=v(\sigma_0(a))\quad\text{for }a\in\mathscr{W}(0).\end{aligned}\tag{2.1}$$

Remark 2.3. Here, F stands for "Frobenius". Note that v_F determines the F-action on $\mathscr{W}$. However, for a given v_F satisfying (2.1), we cannot always find an F-action on $\mathscr{W}$.

The action of $\mathbb{G}_{\mathrm{m}}$ on $\mathscr{W}$ extends to an action on $\mathscr{W}[\hbar^{1/m}]=\mathbf{k}(\hbar^{1/m})\otimes_{\mathbf{k}}\mathscr{W}$ given by $\mathcal{F}_t(\hbar^{1/m})=t\,\hbar^{1/m}$.

Definition 2.4. *A $\mathscr{W}[\hbar^{1/m}]$-module with an* F-*action (or simply a $(\mathscr{W}[\hbar^{1/m}],\mathcal{F})$-module) is a $\mathbb{G}_{\mathrm{m}}$-equivariant $\mathscr{W}[\hbar^{1/m}]$-module: we have isomorphisms $\mathcal{F}_t\colon T_t^{-1}\mathscr{M}\xrightarrow{\sim}\mathscr{M}$ for $t\in\mathbb{C}^\times$ and we assume that*

(a) *$\mathcal{F}_t(u)$ depends holomorphically on t for any $u\in\mathscr{M}$ (i.e., there exist locally finitely many u_i such that $\mathcal{F}_t(u)=\sum_i a_i(t)u_i$ where $a_i(t)\in\mathscr{W}[\hbar^{1/m}]$ depends holomorphically on t),*
(b) *$\mathcal{F}_t(au)=\mathcal{F}_t(a)\mathcal{F}_t(u)$ for $a\in\mathscr{W}[\hbar^{1/m}]$, $u\in\mathscr{M}$,*
(c) *$\mathcal{F}_t\circ\mathcal{F}_{t'}=\mathcal{F}_{tt'}$ for $t,t'\in\mathbb{C}^\times$.*

We denote by $\operatorname{Mod}_F(\mathscr{W}[\hbar^{1/m}])$ the category of $(\mathscr{W}[\hbar^{1/m}],\mathcal{F})$-modules: morphisms are morphisms of $\mathscr{W}[\hbar^{1/m}]$-modules compatible with the $\mathbb{G}_{\mathrm{m}}$-action. We denote by $\operatorname{Mod}^{\mathrm{good}}_F(\mathscr{W}[\hbar^{1/m}])$ its full subcategory of good $(\mathscr{W}[\hbar^{1/m}],\mathcal{F})$-modules. These are $\mathbb{C}$-linear abelian categories.

Note that if there is a relatively compact open subset U of X such that $\mathbb{C}^\times \cdot U = X$, then a good $(\mathscr{W}[\hbar^{1/m}], \mathcal{F})$-module admits a coherent $(\mathscr{W}(0)[\hbar^{1/m}], \mathcal{F})$-lattice.

Assume $X = \{\mathrm{pt}\}$, so that $\mathscr{W} = \mathbf{k}$. We have an equivalence $\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}]) \xrightarrow{\sim} \mathrm{Mod}(\mathbb{C})$, $\mathscr{M} \mapsto \mathscr{M}^{\mathbb{G}_\mathrm{m}}$, with quasi-inverse $V \mapsto \mathbb{C}((\hbar^{1/m})) \otimes_\mathbb{C} V$.

Remark 2.5. Kontsevich and Kaledin [15] have also studied quantization for a symplectic variety with a $\mathbb{G}_\mathrm{m}$-action that stabilizes $\mathbb{C}\omega_X$ with a positive weight.

2.3.2. Let $\mathscr{W}$ be a W-algebra with an F-action with exponent m. Let n be a positive integer and consider the restriction of the F-action via $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m},\ t \mapsto t^n$: we have a new action given by $T'_t = T_{t^n}$ and $\mathcal{F}'_t = \mathcal{F}_{t^n}$. This defines an F-action on $\mathscr{W}$ with exponent mn. Then, we have quasi-inverse equivalences of categories

$$\begin{array}{rcl} \mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}]) & \xleftrightarrow{\sim} & \mathrm{Mod}_F(\mathscr{W}[\hbar^{1/nm}]) \\ \mathscr{M} & \mapsto & \mathscr{W}[\hbar^{1/nm}] \otimes_{\mathscr{W}[\hbar^{1/m}]} \mathscr{M} \\ \left\{ s \in \mathscr{N}\ ; \mathcal{F}'_\zeta(s) = s \text{ for any } \zeta \in \mathbb{C} \text{ with } \zeta^n = 1 \right\} & \leftarrow\!\shortmid & \mathscr{N} \end{array}$$

Remark 2.6. The equivalence above shows the category depends only on the 1-parameter subgroup of $\mathrm{Aut}(X, \mathscr{W})$ given by the $\mathbb{G}_\mathrm{m}$-action.

Let $\hat{\mathbb{G}}_\mathrm{m} = \varprojlim_n \mathbb{G}_\mathrm{m}$, where the limit is taken over maps $f_{n,n'} \colon \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m},\ t \mapsto t^{n/n'}$ for positive integers n, n' with $n'|n$. This is a pro-algebraic group (some sort of universal covering group of $\mathbb{G}_\mathrm{m}$). In terms of functions, we have $\hat{\mathbb{G}}_\mathrm{m} = \mathrm{Spec}(\bigoplus_{a \in \mathbb{Q}} \mathbb{C}t^a)$ with multiplication coming from the coproduct $t^a \mapsto t^a \otimes t^a$. Instead of considering $\mathbb{G}_\mathrm{m}$-actions as above, we could consider $\hat{\mathbb{G}}_\mathrm{m}$-actions on X such that $T_t^* \omega_X = t\omega_X$. Although theoretically more satisfactory, this more complicated formulation is not used in the present paper.

2.3.3. Let us now give two examples.

Let M be a manifold, $X = T^*M$ and $\mathscr{W} = \mathscr{W}_{T^*M}$. We consider the canonical $\mathbb{G}_\mathrm{m}$-action given by $T_t(x, \xi) = (x, t\xi)$. There is a unique F-action with exponent 1 on $\mathscr{W}$ with $\mathcal{F}|_{\mathscr{D}_M} = \mathrm{id}$. Then, for any $\mathbb{G}_\mathrm{m}$-invariant open subset U of X, we have an equivalence

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}|_U) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{good}}(\mathcal{E}_M|_U),\ \mathscr{M} \mapsto \mathscr{M}^{\mathbb{G}_\mathrm{m}}.$$

In particular, we have an equivalence

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_M).$$

Let $X = T^*\mathbb{C}^n$ and $\mathscr{W} = \mathscr{W}_{T^*\mathbb{C}^n}$. Fix $m > 1$ and $l_1, \ldots, l_n \in \{1, \ldots, m-1\}$. We define a $\mathbb{G}_\mathrm{m}$-action by $T_t((x_i), (\xi_i)) = ((t^{l_i} x_i), (t^{m-l_i}\xi_i))$. Then $T_t^*(\omega_X) = t^m \omega_X$. We define an F-action on $\mathscr{W}$ with exponent m by $\mathcal{F}_t(x_i) = t^{l_i} x_i$, $\mathcal{F}_t(\partial_i) = t^{-l_i}\partial_i$, and $\mathcal{F}_t(\hbar) = t^m \hbar$ (note that the relation $[\partial_i, x_i] = 1$ is preserved by $\mathcal{F}_t$). Then,

$$\mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}])}(\mathscr{W}[\hbar^{1/m}])^{\mathrm{opp}} = \mathbb{C}[\hbar^{-l_i/m} x_i, \hbar^{l_i/m}\partial_i; i = 1, \ldots, n] \subset \mathscr{W}[\hbar^{1/m}],$$

which is isomorphic to $\mathscr{D}(\mathbb{C}^n)$. Moreover, $\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}[\hbar^{1/m}])$ is equivalent to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}(\mathbb{C}^n))$ (see Theorem 2.10 below).

2.4. **Equivariance.** We shall discuss G-equivariance of $\mathscr{W}$ by adapting [18, 17] where the $\mathscr{D}$-module version is studied.

2.4.1. Let G be a complex Lie group acting on a symplectic manifold X. Given $g \in G$, let T_g be the corresponding symplectic automorphism of X. Let $\mathfrak{g}$ be the Lie algebra of G and assume that a moment map $\mu_X \colon X \to \mathfrak{g}^*$ is given.

A W-algebra with G-action is a W-algebra with an action of G: we have $\mathbf{k}$-algebra isomorphisms $\rho_g \colon \mathscr{W} \xrightarrow{\sim} T_g^{-1}\mathscr{W}$ for $g \in G$ such that for any $a \in \mathscr{W}$, $\rho_g(a)$ depends holomorphically on $g \in G$. Moreover we assume that there is a *quantized moment map* $\mu_{\mathscr{W}} \colon \mathfrak{g} \to \mathscr{W}(1)$ such that

$$\begin{aligned} &[\mu_{\mathscr{W}}(A), a] = \frac{d}{dt}\rho_{\exp(tA)}(a)|_{t=0}, \\ &\sigma_0(\hbar\mu_{\mathscr{W}}(A)) = A \circ \mu_X, \qquad\qquad \text{for any } A \in \mathfrak{g} \text{ and } a \in \mathscr{W}. \\ &\mu_{\mathscr{W}}(\mathrm{Ad}(g)A) = \rho_g(\mu_{\mathscr{W}}(A)) \end{aligned}$$

Note that $\mu_{\mathscr{W}}$ is a Lie algebra homomorphism.

2.4.2. A quasi-G-equivariant $\mathscr{W}$-module is a $\mathscr{W}$-module $\mathscr{M}$ with an action of G:

$$\rho_g \colon \mathscr{M} \xrightarrow{\sim} T_g^{-1}\mathscr{M}$$

depending holomorphically on $g \in G$ and such that $\rho_g(au) = \rho_g(a)\rho_g(u)$ for $a \in \mathscr{W}$ and $u \in \mathscr{M}$. Then, we have a Lie algebra homomorphism $\alpha \colon \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(\mathscr{M})$ given by $\alpha(A)(u) = \frac{d}{dt}\rho_{\exp(tA)}u|_{t=0}$ for $A \in \mathfrak{g}$ and $u \in \mathscr{M}$. It satisfies

$$\alpha(A)(au) = [\mu_{\mathscr{W}}(A), a]u + a \cdot \alpha(A)(u).$$

It follows that we have a Lie algebra homomorphism

$$\gamma_{\mathscr{M}} \colon \mathfrak{g} \to \mathrm{End}_{\mathscr{W}}(\mathscr{M}),\ A \mapsto \alpha(A) - \mu_{\mathscr{W}}(A). \tag{2.2}$$

The $\mathscr{W}$-module $\mathscr{W}$ is regarded as a quasi-G-equivariant $\mathscr{W}$-module. We have $\alpha(A) = \mathrm{ad}(\mu_{\mathscr{W}}(A))$ and $\gamma_{\mathscr{W}}(A)(a) = -a\mu_{\mathscr{W}}(A)$ ($a \in \mathscr{W}$, $A \in \mathfrak{g}$). Given a G-module V and a quasi-G-equivariant $\mathscr{W}$-module $\mathscr{M}$, the tensor product $\mathscr{M} \otimes V$ has a natural structure of a quasi-G-equivariant $\mathscr{W}$-module. The corresponding γ is given by

$$\gamma_{\mathscr{M}\otimes V}(A)(u \otimes v) = \gamma_{\mathscr{M}}(A)u \otimes v + u \otimes Av \quad \text{for } u \in \mathscr{M},\ v \in V \text{ and } A \in \mathfrak{g}.$$

Let $\lambda \in (\mathfrak{g}^*)^G$. If $\gamma_{\mathscr{M}}$ coincides with the composition $\mathfrak{g} \xrightarrow{\lambda} \mathbb{C} \xrightarrow{z \mapsto z \cdot \mathrm{Id}_{\mathscr{M}}} \mathrm{End}_{\mathscr{W}}(\mathscr{M})$, we say that $\mathscr{M}$ is a twisted G-equivariant $\mathscr{W}$-module with twist λ. For such a coherent module $\mathscr{M}$, we have $\mathrm{Supp}(\mathscr{M}) \subset \mu_X^{-1}(0)$.

We denote by $\mathrm{Mod}(\mathscr{W}, G)$ the category of quasi-G-equivariant $\mathscr{W}$-modules, and by $\mathrm{Mod}^G_\lambda(\mathscr{W})$ its full subcategory of twisted G-equivariant $\mathscr{W}$-modules with twist λ. We denote by $\mathrm{Mod}^{G,\,\mathrm{good}}_\lambda(\mathscr{W})$ the category of good twisted G-equivariant $\mathscr{W}$-modules with twist λ.

The embedding $\mathrm{Mod}^G_\lambda(\mathscr{W}) \to \mathrm{Mod}(\mathscr{W}, G)$ has a left adjoint

$$\begin{aligned} &\Phi_\lambda \colon \mathrm{Mod}(\mathscr{W}, G) \to \mathrm{Mod}^G_\lambda(\mathscr{W}) \\ &\Phi_\lambda(\mathscr{M}) = \mathscr{M} / \big(\sum_{A \in \mathfrak{g}} (\gamma_{\mathscr{M}}(A) - \lambda(A))\mathscr{M}\big). \end{aligned} \tag{2.3}$$

Let V be a one-dimensional G-module and $\chi \in (\mathfrak{g}^*)^G$ its infinitesimal character. Then, we have an equivalence

$$\mathrm{Mod}^G_\lambda(\mathscr{W}) \xrightarrow{\sim} \mathrm{Mod}^G_{\lambda+\chi}(\mathscr{W}),\ \mathscr{M} \mapsto \mathscr{M} \otimes V. \tag{2.4}$$

Let $\mathscr{W}$ be a W-algebra with an F-action with exponent m. A G-action on $(\mathscr{W}, \mathcal{F})$ is a G-action on $\mathscr{W}$ such that T_t and T_g commute, $\mathcal{F}_t$ and $\rho(g)$ commute and $\mu_{\mathscr{W}}(A)$ is $\mathcal{F}_t$-invariant, for $t \in \mathbb{C}^\times$, $g \in G$ and $A \in \mathfrak{g}$.

We define similarly the notion of twisted G-equivariant $(\mathscr{W}[\hbar^{1/m}], \mathcal{F})$-modules. We denote by $\mathrm{Mod}_{F,\lambda}^{G,\,\mathrm{good}}(\mathscr{W}[\hbar^{1/m}])$ the category of good twisted G-equivariant $(\mathscr{W}[\hbar^{1/m}], \mathcal{F})$-modules with twist $\lambda \in (\mathfrak{g}^*)^G$.

2.5. **Symplectic reduction.** Let X be a symplectic manifold with a symplectic action of G and a moment map $\mu_X \colon X \to \mathfrak{g}^*$. *Assume that G acts properly and freely on X* (i.e., the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (gx, x)$ is a closed embedding). Then, $\mu_X^{-1}(0)$ is an involutive submanifold. Let $Z = \mu_X^{-1}(0)/G$, and let $p\colon \mu_X^{-1}(0) \to Z$ be the projection. Then Z carries a natural symplectic structure such that p preserves the symplectic form (i.e., denoting by ω_Z the symplectic form of Z, we have $p^*\omega_Z = \omega_X \mid_{\mu_X^{-1}(0)}$). The local form of X is given by the following Lemma [10, §41].

Lemma 2.7. *Locally on Z, the manifold X is isomorphic to $T^*G \times Z$. More precisely, for any point $x \in \mu_X^{-1}(0)$, there exist a G-invariant open neighbourhood U of x in X and a G-equivariant open symplectic embedding $U \to T^*G \times T^*\mathbb{C}^n$ compatible with the moment maps.*

Let $\mathscr{W}$ be a W-algebra on X with a G-action. Let $\lambda \in (\mathfrak{g}^*)^G$. Set

$$\mathscr{L}_\lambda := \Phi_\lambda(\mathscr{W}) = \mathscr{W} / \sum_{A \in \mathfrak{g}} \mathscr{W}(\mu_{\mathscr{W}}(A) + \lambda(A)).$$

Then, $\mathscr{L}_\lambda$ is a coherent twisted G-equivariant $\mathscr{W}$-module with twist λ. The support of $\mathscr{L}_\lambda$ coincides with $\mu_X^{-1}(0)$. Let $\mathscr{L}_\lambda(0)$ be the $\mathscr{W}(0)$-lattice $\mathscr{W}(0)/\sum_{A \in \mathfrak{g}} \mathscr{W}(-1)(\mu_{\mathscr{W}}(A) + \lambda(A))$ of $\mathscr{L}_\lambda$.

Let $\mathscr{W}_Z = \big((p_* \mathscr{E}nd_{\mathscr{W}}(\mathscr{L}_\lambda))^G\big)^{\mathrm{opp}}$, a sheaf of $\mathbf{k}$-algebras on Z.

Proposition 2.8. (i) *$\mathscr{W}_Z$ is a W-algebra on Z, and $\mathscr{W}_Z(0) \simeq \big((p_* \mathscr{E}nd_{\mathscr{W}(0)}(\mathscr{L}_\lambda(0)))^G\big)^{\mathrm{opp}}$.*

(ii) *We have quasi-inverse equivalences of categories*

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{good}}(\mathscr{W}_Z) & \xleftrightarrow{\sim} & \mathrm{Mod}_\lambda^{G,\,\mathrm{good}}(\mathscr{W}) \\ \mathscr{N} & \mapsto & \mathscr{L}_\lambda \otimes_{p^{-1}\mathscr{W}_Z} p^{-1}\mathscr{N} \\ (p_* \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}))^G & \leftarrow & \mathscr{M}. \end{array}$$

(iii) *Let V be a one-dimensional representation with infinitesimal character χ. Then $\mathscr{N}_{\lambda,\chi}(0) {:=} (p_* \mathscr{H}om_{\mathscr{W}(0)}(\mathscr{L}_\lambda(0), \mathscr{L}_{\lambda-\chi}(0) \otimes V))^G$ is a $\mathscr{W}_Z(0)$-lattice of a coherent $\mathscr{W}_Z$-module $\mathscr{N}_{\lambda,\chi} := (p_* \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{L}_{\lambda-\chi} \otimes V))^G$ and $\mathscr{N}_{\lambda,\chi}(0)/\hbar\mathscr{N}_{\lambda,\chi}(0)$ is isomorphic to $\big(p_*(\mathscr{O}_{\mu_X^{-1}(0)} \otimes V)\big)^G$, the line bundle on Z associated with V.*

(iv) *Assume that $\mathscr{W}$ has an F-action with exponent m compatible with the G-action. Then $\mathscr{W}_Z$ has a natural F-action with exponent m and we have an equivalence of categories:*

$$\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}_Z[\hbar^{1/m}]) \simeq \mathrm{Mod}_{F,\lambda}^{G,\mathrm{good}}(\mathscr{W}[\hbar^{1/m}]).$$

Note that $\mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}) \simeq p^{-1}((p_* \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}))^G)$. Hence, if G is connected, we have $p_* \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}) \simeq (p_* \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_\lambda, \mathscr{M}))^G$.

2.6. **$\mathscr{W}$-affinity.**

2.6.1. Let X be a symplectic manifold. Let S be a variety, let $f\colon X \to S$ be a projective morphism, and let L be a relatively ample line bundle on X. Let $\mathscr{W}$ be a W-algebra on X. The following theorem is an analogue of the result of Beilinson-Bernstein [1] on $\mathscr{D}$-modules on flag manifolds. We follow the formulation of [17].

Theorem 2.9. *For $n > 0$, let $\mathscr{L}_n(0)$ be a locally free $\mathscr{W}(0)$-module of rank 1 such that $\mathscr{L}_n(0)/\hbar\mathscr{L}_n(0) = L^{\otimes(-n)}$. Set $\mathscr{L}_n = \mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{L}_n(0)$.*

Consider the conditions:

(2.5) *for $n \gg 0$, there exists a vector space V_n and a split epimorphism $\mathscr{L}_n \otimes V_n \twoheadrightarrow \mathscr{W}$, i.e., $\mathscr{W}$ is a direct summand of the direct sum of finitely many copies of $\mathscr{L}_n$;*

(2.6) *for $n \gg 0$, there exists a vector space V_n and an epimorphism $\mathscr{W} \otimes V_n \twoheadrightarrow \mathscr{L}_n$.*

(i) *Assume* (2.5). *Then, for every good $\mathscr{W}$-module $\mathscr{M}$, we have $R^i f_*(\mathscr{M}) = 0$ for $i \neq 0$.*

(ii) *Assume* (2.6). *Then, every good $\mathscr{W}$-module is generated by its global sections (locally on S).*

The proof will be given in the next two subsections.

Assume that $\mathscr{W}$ has an F-action with exponent m and that S has a $\mathbb{G}_{\mathrm{m}}$-action such that f is $\mathbb{G}_{\mathrm{m}}$-equivariant. Assume moreover that there exists $\mathrm{o} \in S$ such that every point of S shrinks to o (i.e., $\lim\limits_{t\to 0} tx = \mathrm{o}$ for any $x \in S$).

Let $\widetilde{\mathscr{W}} = \mathscr{W}[\hbar^{1/m}]$ and $A = \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}})^{\mathrm{opp}}$.

Theorem 2.10. *Assume Conditions* (2.5) *and* (2.6) *hold. Then, A is a left noetherian ring and we have quasi-inverse equivalences of categories between $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ and $\mathrm{Mod}_{\mathrm{coh}}(A)$*

$$\begin{array}{rcl} \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}}) & \xleftrightarrow{\sim} & \mathrm{Mod}_{\mathrm{coh}}(A) \\ \mathscr{M} & \mapsto & \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}) \\ \widetilde{\mathscr{W}} \otimes_A M & \leftarrow\!\shortmid & M. \end{array}$$

The proof will be given in § 2.6.4.

2.6.2. *Vanishing theorem.* Let $\mathscr{W}$ be a W-algebra on a symplectic manifold X. Let $\mathscr{M}$ be a coherent $\mathscr{W}$-module. Recall that $\mathscr{M}(0)$ is a $\mathscr{W}(0)$-*lattice* of $\mathscr{M}$ if $\mathscr{M}(0)$ is a coherent $\mathscr{W}(0)$-submodule of $\mathscr{M}$ such that $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{M}(0) \xrightarrow{\sim} \mathscr{M}$.

We start with the following lemma.

Lemma 2.11. *For any coherent $\mathscr{W}(0)$-module $\mathscr{N}$, the canonical map is an isomorphism*

$$\mathscr{N} \xrightarrow{\sim} \varprojlim_m \mathscr{N}/\hbar^m\mathscr{N}. \tag{2.7}$$

Proof. Let us first show that $\mathscr{N} \to \varprojlim\limits_m \mathscr{N}/\hbar^m\mathscr{N}$ is a monomorphism. For any $x \in X$, we have morphisms of $\mathscr{W}(0)_x$-modules:

$$\mathscr{N}_x \to \big(\varprojlim_m \mathscr{N}/\hbar^m\mathscr{N}\big)_x \to \varprojlim_m \big(\mathscr{N}_x/\hbar^m\mathscr{N}_x\big).$$

Since the composition is injective (Artin-Rees argument, see e.g. [25]), the map $\mathscr{N}_x \to \big(\varprojlim_m \mathscr{N}/\hbar^m\mathscr{N}\big)_x$ is injective.

Let us show now that $\mathscr{N} \to \varprojlim_m \mathscr{N}/\hbar^m\mathscr{N}$ is surjective. The question being local, we can take an exact sequence of coherent $\mathscr{W}(0)$-modules

$$0 \to \mathscr{M} \to \mathscr{L} \to \mathscr{N} \to 0,$$

where $\mathscr{L}$ is a free $\mathscr{W}(0)$-module of finite rank. For any Stein open subset U and $m > 0$, we have

$$H^1(U, \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) = 0,$$

and

$$\Gamma(U, \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) \to \Gamma(U, \mathscr{M}/(\hbar^{m-1}\mathscr{L} \cap \mathscr{M})) \text{ is surjective.}$$

Indeed, in the exact sequence

$$\begin{aligned}\Gamma(U; \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) &\to \Gamma(U; \mathscr{M}/(\hbar^{m-1}\mathscr{L} \cap \mathscr{M}))\\ &\to H^1(U; (\hbar^{m-1}\mathscr{L} \cap \mathscr{M})/(\hbar^m\mathscr{L} \cap \mathscr{M}))\\ &\to H^1(U; \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) \to H^1(U; \mathscr{M}/(\hbar^{m-1}\mathscr{L} \cap \mathscr{M})),\end{aligned}$$

$H^1(U; (\hbar^{m-1}\mathscr{L} \cap \mathscr{M})/(\hbar^m\mathscr{L} \cap \mathscr{M}))$ vanishes because $(\hbar^{m-1}\mathscr{L} \cap \mathscr{M})/(\hbar^m\mathscr{L} \cap \mathscr{M})$ is a coherent $\mathscr{O}_X$-module.

It follows that the following sequence is exact

$$0 \to \Gamma(U, \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) \to \Gamma(U, \mathscr{L}/\hbar^m\mathscr{L}) \to \Gamma(U, \mathscr{N}/\hbar^m\mathscr{N}) \to 0.$$

Since $\{\Gamma(U, \mathscr{M}/(\hbar^m\mathscr{L}\cap\mathscr{M}))\}_m$ satisfies the ML condition, the bottom row of the following commutative diagram is exact

$$\begin{tikzcd}
& & \Gamma(U,\mathscr{L}) \arrow[r] \arrow[d, "\sim"] & \Gamma(U,\mathscr{N}) \arrow[d] & \\
0 \arrow[r] & \varprojlim_m \Gamma(U, \mathscr{M}/(\hbar^m\mathscr{L} \cap \mathscr{M})) \arrow[r] & \varprojlim_m \Gamma(U, \mathscr{L}/\hbar^m\mathscr{L}) \arrow[r] & \varprojlim_m \Gamma(U, \mathscr{N}/\hbar^m\mathscr{N}) \arrow[r] & 0.
\end{tikzcd}$$

It follows that $\Gamma(U, \mathscr{N}) \to \varprojlim_m \Gamma(U, \mathscr{N}/\hbar^m\mathscr{N}) \simeq \Gamma(U, \varprojlim_m \mathscr{N}/\hbar^m\mathscr{N})$ is surjective. $\square$

Lemma 2.12. *Let $\mathscr{M}$ be a coherent $\mathscr{W}$-module and let $\mathscr{M}(0)$ be a $\mathscr{W}(0)$-lattice of $\mathscr{M}$. Set $\mathscr{M}(m) = \hbar^{-m}\mathscr{M}(0)$ and $\overline{\mathscr{M}} = \mathscr{M}(0)/\mathscr{M}(-1)$. Assume that*

$$H^i(X, \overline{\mathscr{M}}) = 0 \text{ for } i \neq 0.$$

Then,

(i) *the canonical morphism*

$$\Gamma(X, \mathscr{M}(0))/\Gamma(X, \mathscr{M}(-m)) \longrightarrow \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m))$$

is an isomorphism for any $m \geqslant 0$,

(ii) *$H^i(X, \mathscr{M}(0)) = 0$ for any $i \neq 0$.*

Proof. Given $m \geqslant 0$, the exact sequence

$$0 \to \overline{\mathscr{M}} \xrightarrow{\hbar^m} \mathscr{M}(0)/\mathscr{M}(-m-1) \to \mathscr{M}(0)/\mathscr{M}(-m) \to 0$$

induces exact sequences

$$\Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m-1)) \to \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m)) \to H^1(X, \overline{\mathscr{M}})$$

and

$$H^i(X, \overline{\mathscr{M}}) \to H^i(X, \mathscr{M}(0)/\mathscr{M}(-m-1)) \to H^i(X, \mathscr{M}(0)/\mathscr{M}(-m)).$$

It follows that $\Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m-1)) \to \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m))$ is surjective for any $m \geqslant 0$ and $H^i(X, \mathscr{M}(0)/\mathscr{M}(-m)) = 0$ for any $i > 0$. Since $\Gamma(X, \mathscr{M}(0)) = \varprojlim_m \Gamma(X, \mathscr{M}(0)/\mathscr{M}(-m))$ by Lemma 2.11, we obtain (i).

For $i > 0$, we have

$$H^i(X, \mathscr{M}(0)) = \varprojlim_m H^i(X, \mathscr{M}(0)/\mathscr{M}(-m)) = 0$$

because $\{H^{i-1}(X, \mathscr{M}(0)/\mathscr{M}(-m))\}_m$ satisfies the ML condition. □

2.6.3. *Proof of Theorem 2.9.* Let us prove (i). The question being local on S, we may assume that there exists a $\mathscr{W}(0)$-lattice $\mathscr{M}(0)$ of $\mathscr{M}$. Set $\overline{\mathscr{M}} = \mathscr{M}(0)/\hbar\mathscr{M}(0)$. Then, for $m \gg 0$, we have $R^i f_*(L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}) = 0$ for $i \neq 0$. It follows that

$$H^i(f^{-1}U, L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}) = 0$$

for any $i \neq 0$ and any Stein open subset U of S. From now on, we assume that m is large enough so that the vanishing above holds.

Let $\mathscr{A}_m = \mathscr{E}nd_{\mathscr{W}}(\mathscr{L}_m)^{\mathrm{opp}}$, a W-algebra on X. We have $\mathscr{A}_m(0) = \mathscr{E}nd_{\mathscr{W}(0)}(\mathscr{L}_m(0))^{\mathrm{opp}}$. Let $\mathscr{L}_m(0)^* = \mathscr{H}om_{\mathscr{W}(0)}(\mathscr{L}_m(0), \mathscr{W}(0))$, an $(\mathscr{A}_m(0), \mathscr{W}(0))$-bimodule, and let $\mathscr{L}_m^* = \mathscr{H}om_{\mathscr{W}}(\mathscr{L}_m, \mathscr{W})$, an $(\mathscr{A}_m, \mathscr{W})$-bimodule. We have

$$\mathscr{L}_m^* \simeq \mathscr{A}_m \otimes_{\mathscr{A}_m(0)} \mathscr{L}_m(0)^* \simeq \mathscr{L}_m(0)^* \otimes_{\mathscr{W}(0)} \mathscr{W}.$$

Note that the bimodules $\mathscr{L}_m$ and $\mathscr{L}_m^*$ give inverse Morita equivalences between $\mathscr{A}_m$ and $\mathscr{W}$.

Let $\mathscr{M}_m(0) = \mathscr{L}_m^*(0) \otimes_{\mathscr{W}(0)} \mathscr{M}(0)$, an $\mathscr{A}_m(0)$-lattice in the $\mathscr{A}_m$-module $\mathscr{M}_m = \mathscr{L}_m^* \otimes_{\mathscr{W}} \mathscr{M}$. We have $\mathscr{M}_m(0)/\hbar\mathscr{M}_m(0) \simeq L^{\otimes m} \otimes_{\mathscr{O}_X} \overline{\mathscr{M}}$, hence $H^i(f^{-1}U, \mathscr{M}_m(0)/\hbar\mathscr{M}_m(0)) = 0$ for $i \neq 0$. Lemma 2.12 (ii) implies that $H^i(f^{-1}U, \mathscr{M}_m(0)) = 0$ for $i \neq 0$. Taking the inductive limit with respect to Stein open neighbourhoods U of $s \in S$, we obtain $H^i(f^{-1}(s), \mathscr{M}_m(0)) = 0$, hence

$$H^i(f^{-1}(s), \mathscr{M}_m) \simeq \mathbf{k} \otimes_{\mathbf{k}(0)} H^i(f^{-1}(s), \mathscr{M}_m(0)) = 0. \tag{2.8}$$

By Condition (2.5), $\mathscr{W}$ is a direct summand of a direct sum of finitely many copies of the left $\mathscr{W}$-module $\mathscr{L}_m$. So, $\mathscr{W}$ is a direct summand of a direct sum of finitely many copies of the right $\mathscr{W}$-module $\mathscr{L}_m^*$ and $\mathscr{M}$ is a direct summand of a direct sum of finitely many copies of $\mathscr{M}_m$ (as a sheaf). Then, (2.8) implies that $H^i(f^{-1}(s), \mathscr{M}) = 0$. This completes the proof of (i).

We now prove (ii). We shall keep the same notations as in the proof of (i). Since L is relatively ample, given $s \in S$, there exists a surjective map $(\mathscr{O}_X |_{f^{-1}(s)})^{\oplus N} \twoheadrightarrow (L^{\otimes m} \otimes \overline{\mathscr{M}}) |_{f^{-1}(s)}$ for some N. On the other hand, Lemma 2.12 (i) implies that

$\Gamma(f^{-1}(s), \mathscr{M}_m(0)) \to \Gamma(f^{-1}(s), \mathscr{M}_m(0)/\hbar\mathscr{M}_m(0))$ is surjective. Hence we have a morphism $\phi_m\colon \mathscr{A}_m(0)^{\oplus N}|_{f^{-1}(s)} \to \mathscr{M}_m(0)|_{f^{-1}(s)}$ such that the composition $\mathscr{A}_m(0)^{\oplus N}|_{f^{-1}(s)} \to (\mathscr{M}_m(0)/\hbar\mathscr{M}_m(0))|_{f^{-1}(s)}$ is an epimorphism. It follows that ϕ_m is an epimorphism. Thus, there exists an epimorphism $\mathscr{A}_m^{\oplus N}|_{f^{-1}(s)} \twoheadrightarrow \mathscr{M}_m|_{f^{-1}(s)}$. By applying the exact functor $\mathscr{L}_m \otimes_{\mathscr{A}_m} \bullet\colon \mathrm{Mod}(\mathscr{A}_m) \to \mathrm{Mod}(\mathscr{W})$, we obtain an epimorphism $\mathscr{L}_m^{\oplus N}|_{f^{-1}(s)} \twoheadrightarrow \mathscr{M}|_{f^{-1}(s)}$. The assertion follows now from Condition (2.6).

2.6.4. *Proof of Theorem 2.10.* By Theorem 2.9, $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}}) \ni \mathscr{M} \mapsto f_*(\mathscr{M}) \in \mathrm{Mod}(f_*(\widetilde{\mathscr{W}}))$ is an exact functor.

By the assumption, o has a neighbourhood system consisting of relatively compact Stein open neighbourhoods U such that U is stable by T_t $(0 < |t| \leqslant 1)$. For such an U, we have $S = \bigcup_{t\in\mathbb{C}^*} T_t U$. For any $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$, we have

$$\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}) = \left\{ s \in \mathscr{M}(f^{-1}U)\,;\, s \text{ is F-invariant} \right\}.$$

Here $s \in \mathscr{M}(f^{-1}U)$ is F-invariant if $\mathcal{F}_t(s) = s$ for any $t \in \mathbb{C}^\times$ with $|t| = 1$.

For $s \in \mathscr{M}(f^{-1}U)$, let

$$p_n(s) = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=1} t^{-n}\mathcal{F}_t(s)\frac{dt}{t}.$$

We have $s = \sum_n p_n(s)$ and $\hbar^{-n/m}p_n(s) = p_0(\hbar^{-n/m}s)$ is F-invariant.

Lemma 2.13. $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \bullet)$ *is an exact functor.*

Proof. Let $\varphi\colon \mathscr{M} \to \mathscr{M}' \to 0$ be an epimorphism in $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ and let $s' \in \mathscr{M}'(f^{-1}U)$ such that $\mathcal{F}_t(s') = s'$ for any t with $|t| = 1$. By Theorem 2.9, there exists $s \in \mathscr{M}(f^{-1}U)$ such that $\varphi(s) = s'$. We have $\varphi(p_0(s)) = s'$ and $p_0(s)$ is F-invariant. □

Lemma 2.14. *Any* $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ *is generated by F-invariant global sections.*

Proof. By Theorem 2.9, $\mathscr{M}$ is generated by global sections $s_i \in \mathscr{M}(f^{-1}U)$. Then, $\mathscr{M}$ is generated by the $\hbar^{-n/m}p_n(s_i)$'s. Indeed, let $\mathscr{N}$ be the submodule of $\mathscr{M}$ generated by the $p_n(s_i)$'s. This is a coherent submodule of $\mathscr{M}$. Let $\psi\colon \mathscr{M} \to \mathscr{M}/\mathscr{N}$ be the quotient morphism. Then $p_n\psi(s_i) = \psi(p_n(s_i)) = 0$ for any n, and hence $\psi(s_i) = 0$. It follows that $\mathscr{N} = \mathscr{M}$. □

We deduce that $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M})$ is an A-module of finite presentation for any $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$.

Lemma 2.15. *A is left noetherian.*

Proof. Let I be a left ideal of A. Let $\mathscr{I} \subset \widetilde{\mathscr{W}}$ be the image of $\widetilde{\mathscr{W}} \otimes_A I \to \widetilde{\mathscr{W}}$. Note that $\mathscr{I}$ belongs to $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$. Since $\widetilde{\mathscr{W}}$ is coherent, there exist finitely many $a_i \in I$ such that $\mathscr{I} = \sum \widetilde{\mathscr{W}} a_i$. We have $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{I}) = \sum_i Aa_i \subset I$ by Lemma 2.13. Since we have injective maps $I \to \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{I}) \hookrightarrow \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \widetilde{\mathscr{W}}) = A$, we obtain $I = \sum_i Aa_i$. □

Since good $(\widetilde{\mathscr{W}}, F)$-modules are generated by F-invariant sections, $\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \bullet)$ sends $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ to $\mathrm{Mod}_{\mathrm{coh}}(A)$.

Given $M \in \mathrm{Mod}_{\mathrm{coh}}(A)$, the canonical morphism

$$M \to \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \widetilde{\mathscr{W}} \otimes_A M)$$

is an isomorphism because both sides are right exact functors of M and the morphism is an isomorphism for $M = A$.

Given $\mathscr{M} \in \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$, the canonical map $\widetilde{\mathscr{W}} \otimes_A \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})}(\widetilde{\mathscr{W}}, \mathscr{M}) \to \mathscr{M}$ is an isomorphism, because both sides are right exact functors of $\mathscr{M}$ and $\mathscr{M}$ has a resolution $\widetilde{\mathscr{W}}^{\oplus m_1} \to \widetilde{\mathscr{W}}^{\oplus m_0} \to \mathscr{M} \to 0$ in $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{W}})$ by Lemma 2.14.

This completes the proof of Theorem 2.10.

3. Rational Cherednik algebras and $\mathscr{D}$-modules

3.1. **Definitions, notations and recollections.**

3.1.1. Let $V = \mathbb{C}^n$, let $G = \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ and let $\mathfrak{g} = \mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$. We denote by $e_{rs} \in \mathfrak{g}$ the elementary matrix with 0 coefficients everywhere except in row r and column s where the coefficient is 1. We denote by $A_{rs} \in \mathbb{C}[\mathfrak{g}]$ the corresponding coordinate function.

We denote by $\mathfrak{t} = \mathbb{C}^n$ the Cartan subalgebra of diagonal matrices of $\mathfrak{g}$ and by $W = S_n$ the Weyl group. We denote by s_{ij} the transposition (ij) for $1 \leqslant i \neq j \leqslant n$. We have $\mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\mathfrak{t}^*] = \mathbb{C}[y_1, \ldots, y_n]$.

We put $\mathfrak{d}(x) = \prod_{i<j}(x_i - x_j) \in \mathbb{C}[\mathfrak{t}]$. We denote by $\mathfrak{g}_{\mathrm{reg}}$ the open subset of regular semisimple elements of $\mathfrak{g}$ and we put $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}} = \{x \in \mathfrak{t}; \mathfrak{d}(x) \neq 0\}$.

We will identify $\mathbb{C}[\mathfrak{t}]^W$ and $\mathbb{C}[\mathfrak{g}]^G$ via the restriction map.

Given M a graded vector space, we denote by M_k its component of degree k.

3.1.2. Let X be a manifold, $i\colon Y \hookrightarrow X$ a submanifold, and let $f\colon \mathscr{M} \to \mathscr{N}$ be a morphism of coherent $\mathscr{D}_X$-modules. Assume Y is non-characteristic for $\mathscr{M}$ and $\mathscr{N}$ (i.e., for $Z = \mathrm{Ch}(\mathscr{M})$ or $Z = \mathrm{Ch}(\mathscr{N})$, we have $Z \cap T_Y^*X \subset T_X^*X$). If $i^*(f)\colon i^*\mathscr{M} \to i^*\mathscr{N}$ is an isomorphism (resp. monomorphism, epimorphism), then so is f on a neighbourhood of Y (see e.g. [19, Theorem 4.7]).

3.1.3. Let $f \in H^0(X; \mathscr{O}_X)$ be non zero. We denote by $\delta(f)$ the element f^{-1} of the $\mathscr{D}_X$-module $\mathscr{O}_X[f^{-1}]/\mathscr{O}_X$. So, $\mathscr{D}_X\delta(f) \subset \mathscr{O}_X[f^{-1}]/\mathscr{O}_X$. More generally, let S be a closed subvariety of complete intersection of codimension r given by $f_1 = \cdots = f_r = 0$ for $f_1, \ldots, f_r \in H^0(X; \mathscr{O}_X)$. Then

$$\mathscr{H}_S^j(\mathscr{O}_X) = 0 \text{ for } j \neq r \text{ and } \mathscr{H}_S^r(\mathscr{O}_X) \simeq \mathscr{O}[(f_1 \cdots f_r)^{-1}]/\sum_{1 \leqslant i \leqslant r} \mathscr{O}[(f_1 \cdots \hat{f}_i \cdots f_r)^{-1}].$$

We denote the last $\mathscr{D}_X$-module by $\mathscr{B}_{S|X}$. We denote by $\delta(f_1)\cdots\delta(f_r)$ the section $1/(f_1 \cdots f_r)$ of $\mathscr{B}_{S|X}$.

3.2. **Construction of some $\mathscr{D}$-modules.**

3.2.1. Given $c \in \mathbb{C}$, we denote by H_c the rational Cherednik algebra of $(\mathfrak{t}, W)$ with parameter c: this is the $\mathbb{C}$-algebra quotient of $T(\mathfrak{t}^* \oplus \mathfrak{t}) \rtimes W$ by the relations

$$\begin{aligned}
&[x_i, x_j] = 0, \quad [y_i, y_j] = 0,\\
&[y_i, x_j] = c s_{ij} \quad \text{for } i \neq j,\\
&[y_i, x_i] = 1 - c \sum_{k \neq i} s_{ik}.
\end{aligned}$$

We have a vector space decomposition ("PBW-property") [6, Theorem 1.3]

$$H_c = \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}^*] \otimes \mathbb{C}[W].$$

There is an injective algebra morphism (given by Dunkl operators) [6, Proposition 4.5]

$$\theta_c \colon H_c \hookrightarrow \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}])$$

given by the canonical map on $\mathbb{C}[\mathfrak{t}] \rtimes W$ and by

$$\theta_c(y_i) = \partial_{x_i} - c \sum_{k \neq i} \frac{1}{x_i - x_k}(1 - s_{ik}). \tag{3.1}$$

It induces an isomorphism of algebras after localization

$$\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{t}]} H_c \xrightarrow{\sim} \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W.$$

We denote by $e := \frac{1}{n!} \sum_{w \in W} w \in \mathbb{C}[W] \subset H_c$ and $e_{\det} := \frac{1}{n!} \sum_{w \in W} \det(w) w \in \mathbb{C}[W] \subset H_c$ the idempotents corresponding to the trivial representation and the sign representation of W.

We have an injective morphism $\mathbb{C}[\mathfrak{t}]^W \to eH_ce$, $a \mapsto ae$, and we identify $\mathbb{C}[\mathfrak{t}]^W$ with its image. We put $\mathbf{y}^2 = \sum_{i=1}^n y_i^2 \in H_c$. Recall that eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W e$ and $\mathbb{C}[\mathfrak{t}^*]^W e$ (cf. e.g. [4, proof of Proposition 5.4.4]). On the other hand, we have an isomorphism of $\mathbb{C}[W]$-modules (cf. e.g. [2, Corollary 4.9])

$$\big(\mathrm{ad}(\mathbf{y}^2)\big)^k \colon \mathbb{C}[\mathfrak{t}]_k \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}^*]_k. \tag{3.2}$$

It sends $a(x_1, \dots, x_n)$ to $2^k k! a(y_1, \dots, y_n)$. Hence eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W e$ and $\mathbf{y}^2 e$.

We denote by $h \mapsto h^*$ the anti-involution of H_c given by $x_i \mapsto x_i$, $y_i \mapsto -y_i$, $w \mapsto w^{-1}$ ($w \in W$).

3.2.2. We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the G-invariant bilinear symmetric form $\mathfrak{g} \times \mathfrak{g} \ni (A, A') \mapsto \mathrm{tr}(AA')$.

A pair (A, z) will denote a point of $\mathfrak{g} \times V$. We identify $T^*(\mathfrak{g} \times V)$ with $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$, and denote accordingly a point in $T^*(\mathfrak{g} \times V)$ by (A, B, z, ζ). Let $\mu \colon T^*(\mathfrak{g} \times V) \to \mathfrak{g}^*$ be the moment map. It is given by $\mu(A, B, z, \zeta) = -[A, B] - z \circ \zeta$.

Let us denote by

$$\mu_D \colon \mathfrak{g} \to \mathscr{D}_{\mathfrak{g} \times V}(\mathfrak{g} \times V)$$

the Lie algebra homomorphism associated with the diagonal action of G on $\mathfrak{g} \times V$. Let us consider the $\mathscr{D}_{\mathfrak{g} \times V}$-module $\mathscr{L}_c = \mathscr{D}_{\mathfrak{g} \times V} u_c$ given by the defining equation:

$$(\mu_D(C) + c\,\mathrm{tr}(C)) u_c = 0 \quad (C \in \mathfrak{g}).$$

More formally, we have $\mathscr{L}_c = \mathscr{D}_{\mathfrak{g} \times V} / (\mathscr{D}_{\mathfrak{g} \times V}(\mu_D + c\,\mathrm{tr})(\mathfrak{g}))$ and u_c is the image of 1 in $\mathscr{L}_c$.

We consider $\mathscr{L}_c$ as a twisted G-equivariant $\mathscr{D}_{\mathfrak{g}\times V}$-module with twist $c\,\mathrm{tr}$, where u_c is a G-invariant section of $\mathscr{L}_c$. Since any $a \in \mathbb{C}[\mathfrak{g}]^G$ commutes with $\mu_D(C)$ ($C \in \mathfrak{g}$), the map $u_c \mapsto au_c$ extends to a $\mathscr{D}_{\mathfrak{g}\times V}$-linear endomorphism of $\mathscr{L}_c$. Hence, $\mathscr{L}_c$ has a $(\mathbb{C}[\mathfrak{t}]^W \otimes \mathscr{D}_{\mathfrak{g}\times V})$-module structure.

The characteristic variety $\mathrm{Ch}(\mathscr{L}_c)$ of $\mathscr{L}_c$ is the almost commuting variety:

$$\mathrm{Ch}(\mathscr{L}_c) = \mu^{-1}(0) = \{(A, B, z, \zeta)\,;\, [A, B] + z \circ \zeta = 0\}\,.$$

This is a complete intersection in $T^*(\mathfrak{g} \times V)$ [7, Theorem 1.1].

Lemma 3.1. *Let $\mathfrak{g}_1$ be the open subset of $\mathfrak{g}$ of elements which have at least $(n-1)$ distinct eigenvalues. We have*

$$\mathscr{H}^0_{(\mathfrak{g}\setminus\mathfrak{g}_{\mathrm{reg}})\times V}(\mathscr{L}_c) = 0 \text{ and } \mathscr{H}^1_{(\mathfrak{g}\setminus\mathfrak{g}_1)\times V}(\mathscr{L}_c) = 0.$$

Proof. Since $\mathrm{Ch}(\mathscr{L}_c)$ is a complete intersection, we have ([19, (2.23)])

$$\mathscr{E}xt^j_{\mathscr{D}_{\mathfrak{g}\times V}}(\mathscr{L}_c, \mathscr{D}_{\mathfrak{g}\times V}) = 0 \text{ for } j \neq \mathrm{codim}_{T^*(\mathfrak{g}\times V)}\mu^{-1}(0) = n^2. \tag{3.3}$$

Let $\gamma\colon \mathfrak{g} \to \mathfrak{t}/W$ be the canonical map associating to $A \in \mathfrak{g}$ the eigenvalues of A. Let $\tilde{\gamma}\colon \mu^{-1}(0) \to \mathfrak{t}/W$ be given by $(A, B, i, j) \mapsto \gamma(A)$. Then, $\tilde{\gamma}$ is a flat morphism [7, Corollary 2.7].

Let S be a closed subset of $\mathfrak{t}/W$. Since $\tilde{\gamma}$ is flat, we have

$$\mathrm{codim}_{T^*(\mathfrak{g}\times V)}(\gamma^{-1}(S) \times_{\mathfrak{g}} \mathrm{Ch}(\mathscr{L}_c)) - \mathrm{codim}_{T^*(\mathfrak{g}\times V)} \mathrm{Ch}(\mathscr{L}_c) = \mathrm{codim}_{\mathfrak{t}/W} S.$$

Lemma 2.1 applied to $\gamma^{-1}(S) \times_{\mathfrak{g}} \mathrm{Ch}(\mathscr{L}_c)$ implies

$$\mathscr{H}^j_{\gamma^{-1}(S)\times V}(\mathscr{L}_c) = 0 \quad \text{for } j < \mathrm{codim}_{\mathfrak{t}/W} S$$

and the lemma follows. $\square$

3.2.3. Let us recall some constructions and results of [14]. Let $\mu_0\colon \mathfrak{g} \to \mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}(\mathfrak{t} \times \mathfrak{g})$ be the morphism given by the action of G on $\mathfrak{t} \times \mathfrak{g}$: $g \cdot (x, A) = (x, \mathrm{Ad}(g)A)$. We consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}$-module generated by $\delta_0(x, A)$ with the defining equations:

$$\mu_0(C)\delta_0(x, A) = 0 \quad \text{for any } C \in \mathfrak{g},$$

$$\left.\begin{array}{rcl} (P(A) - P(x))\delta_0(x, A) & = & 0 \\ (P(\partial_A) - P(-\partial_x))\delta_0(x, A) & = & 0 \end{array}\right. \quad \text{for any } P \in \mathbb{C}[\mathfrak{g}]^G.$$

Then, $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A)$ is a simple holonomic $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}$-module with support $\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{g}$. Its characteristic variety is the set of (x, y, A, B) such that $[A, B] = 0$ and there exists $g \in G$ such that $\mathrm{Ad}(g)A$ and $\mathrm{Ad}(g)B$ are upper triangular and x and y are the diagonal components of $\mathrm{Ad}(g)A$ and $\mathrm{Ad}(g)B$. Note that $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A) \subset \mathscr{B}_{\mathfrak{t}\times_{\mathfrak{t}/W}\mathfrak{g}|\mathfrak{t}\times\mathfrak{g}}$ by $\delta_0(x, A) \mapsto \prod_{i=1}^n \delta(P_i(x) - P_i(A))$ (see § 3.1.3), where $P_i \in \mathbb{C}[\mathfrak{g}]^G$ $(i = 1, \ldots, n)$ are the fundamental invariants given by $\det(1 + tA) = \sum_{i=0}^n P_i(A)t^i$.

We will need to consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A)\boxtimes\mathscr{O}_V$, generated by $\delta(x, A) := \delta_0(x, A)\boxtimes 1$ which satisfies the same equations as $\delta_0(x, A)$ and $\partial_{z_i}\delta(x, A) = 0$. In particular, $\mu_D(C)\delta(x, A) = 0$ for any $C \in \mathfrak{g}$.

3.2.4. Let us set

$$q(A,z)=\det(A^{n-1}z,A^{n-2}z,\dots,Az,z).$$

We have $q(\mathrm{Ad}(g)A,gz)=\det(g)q(A,z)$ for $g\in G$ and $[\mu_D(C),q(A,z)]=-\operatorname{tr}(C)q(A,z)$ for $C\in\mathfrak{g}$.

Consider the $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}q(A,z)^c\delta(x,A)$. A precise definition is as follows. Let us consider the left ideal $\mathscr{I}$ of $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}\otimes\mathbb{C}[s]$ (s being an indeterminate) consisting of those $P(s)$ such that $P(m)q(A,z)^m\delta(x,A)=0$ for any $m\in\mathbb{Z}_{\geqslant 0}$. We now define $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}q(A,z)^c\delta(x,A)$ as $\big(\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}\otimes\mathbb{C}[s]\big)/\big(\mathscr{I}+\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}\otimes\mathbb{C}[s](s-c)\big)$. It is a holonomic $\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}\times V}$-module.

The element $q(A,z)^c\delta(x,A)$ satisfies

$$(\mu_D(C)+c\operatorname{tr}(C))q(A,z)^c\delta(x,A)=0\quad\text{for any } C\in\mathfrak{g},$$
$$(P(A)-P(x))q(A,z)^c\delta(x,A)=0\quad\text{for any } P\in\mathbb{C}[\mathfrak{g}]^G.$$

We put $v_c=q(A,z)^c\delta(x,A)$. Let $p_0\colon\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V\to\mathfrak{g}\times V$ be the projection. Let us consider the $\mathscr{D}_{\mathfrak{g}\times V}$-module

$$\mathscr{M}_c=(p_0)_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}v_c)=(p_0)_*(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}q(A,z)^c\delta(x,A)).$$

By the definition, we have an isomorphism $\mathscr{M}_c\xrightarrow{\sim}j_*j^{-1}\mathscr{M}_c$ where $j\colon\mathfrak{g}_{\mathrm{reg}}\times V\hookrightarrow\mathfrak{g}\times V$ is the open embedding. This is a quasi-coherent $\mathscr{D}_{\mathfrak{g}\times V}$-module whose characteristic variety is contained in the almost commuting variety $\mu^{-1}(0)$.

The action of W on $\mathfrak{t}_{\mathrm{reg}}$ induces a W-action on $\mathscr{M}_c$. Here, W acts trivially on v_c. Hence, the $\mathscr{D}_{\mathfrak{g}\times V}$-module $\mathscr{M}_c$ has a module structure over $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\rtimes W$. Therefore, H_c acts on $\mathscr{M}_c$ via the canonical embedding $\theta_c\colon H_c\hookrightarrow\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\rtimes W$.

3.3. Spherical constructions and shift.

3.3.1. There is a $\mathscr{D}_{\mathfrak{g}\times V}$-linear homomorphism

$$\iota\colon\mathscr{L}_c\to\mathscr{M}_c,\ u_c\mapsto v_c. \tag{3.4}$$

We regard $\mathscr{M}_c$ as a twisted G-equivariant $\mathscr{D}_{\mathfrak{g}\times V}$-module with twist $c\operatorname{tr}$, where sections in $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})v_c$ are G-invariant. Then, the morphism above is G-equivariant. Moreover, it is $\mathbb{C}[\mathfrak{t}]^W$-linear. Hence ι induces an epimorphism of $(\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\rtimes W)\otimes\mathscr{D}_{\mathfrak{g}\times V}$-modules:

$$\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{t}]^W}\mathscr{L}_c\twoheadrightarrow\mathscr{M}_c.$$

Lemma 3.2. *The morphism of $\mathbb{C}[W]\otimes\mathscr{D}_{\mathfrak{g}\times V}$-modules*

$$1\otimes\iota\colon\mathbb{C}[\mathfrak{t}]\otimes_{\mathbb{C}[\mathfrak{t}]^W}\mathscr{L}_c\to\mathscr{M}_c$$

is an isomorphism on $\mathfrak{g}_{\mathrm{reg}}\times V$.

In particular, the induced morphisms $\mathscr{L}_c\xrightarrow{u_c\mapsto v_c}e\mathscr{M}_c$ and $\mathscr{L}_c\xrightarrow{u_c\mapsto\mathfrak{d}(x)v_c}e_{\det}\mathscr{M}_c$ are isomorphisms on $\mathfrak{g}_{\mathrm{reg}}\times V$.

Proof. Let $i\colon\mathfrak{t}_{\mathrm{reg}}\times V\hookrightarrow\mathfrak{g}\times V$ be the embedding. Note that i is non-characteristic for $\mathscr{L}_c$ and $\mathscr{M}_c$. Since $G\cdot\mathfrak{t}_{\mathrm{reg}}=\mathfrak{g}_{\mathrm{reg}}$, it is enough to prove that the canonical map $\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]\otimes_{\mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W}i^*\mathscr{L}_c\to i^*\mathscr{M}_c$ is an isomorphism (cf. § 3.1.2).

We have $i^*\mu_D(e_{rs})=(A_{rr}-A_{ss})\partial_{A_{rs}}-z_s\partial_{z_r}$. It follows that we have an isomorphism

$$\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}/\big(\sum_i\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}(z_i\partial_{z_i}-c)\big)\xrightarrow{\sim}i^*\mathscr{L}_c,\ 1\mapsto i^*u_c.$$

Let $i'' \colon \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}} \hookrightarrow \mathfrak{t} \times \mathfrak{g}$ be the embedding. Since the Jacobian

$$\partial(P_1(x), \ldots, P_n(x))/\partial(x_1, \ldots, x_n)$$

is equal to $\mathfrak{d}(x)$ (e.g. [5, Ch. V, § 5.4, Proposition 5]), we have an isomorphism

$$i''^{*}\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta_0(x, A) \xrightarrow[\sim]{\delta_0(x,A)\mapsto\sum_w \mathfrak{d}(a)^{-1}\delta(w^{-1}x-a)} \bigoplus_{w\in W} \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{t}_{\mathrm{reg}}}\delta(w^{-1}x - a)$$

where $\delta(w^{-1}x - a) = \delta(x_{w(1)} - a_1)\cdots\delta(x_{w(n)} - a_n)$.

Let us denote by $i' \colon \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}} \times V \hookrightarrow \mathfrak{t}_{\mathrm{reg}} \times \mathfrak{g} \times V$ the embedding. We have an isomorphism

$$i'^{*}\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{g}\times V}v_c \xrightarrow[\sim]{v_c\mapsto\sum_w v'_w} \bigoplus_{w\in W} \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{t}_{\mathrm{reg}}\times V}v'_w. \tag{3.5}$$

where $v'_w = \mathfrak{d}(a)^{c-1}(z_1\cdots z_n)^c\delta(w^{-1}x - a)$ has the defining equations

$$\begin{aligned}&\bigl(\partial_{x_{w(i)}} + \partial_{a_i} - (c-1)\textstyle\sum_{j\neq i}\dfrac{1}{a_i - a_j}\bigr)v'_w = 0,\\ &(x_{w(i)} - a_i)v'_w = 0,\\ &(z_i\partial_{z_i} - c)v'_w = 0,\end{aligned} \qquad \text{for any } i = 1, \ldots, n.$$

In particular, we have

$$f(x)v'_w = (w^{-1}f)(a)v'_w \quad \text{for any } f \in \mathbb{C}[\mathfrak{t}]. \tag{3.6}$$

We obtain finally an isomorphism

$$i^{*}\mathscr{M}_c \xrightarrow[\sim]{v_c\mapsto\sum_w v'_w} \bigoplus_{w\in W} \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}v'_w.$$

This is compatible with the action of W, where $w'(v'_w) = v'_{w'w}$. Moreover, each $\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V}v'_w$ is isomorphic to $i^{*}\mathscr{L}_c$ by $v'_w \mapsto u_c$. Hence we obtain an isomorphism of $(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V} \otimes \mathbb{C}[W])$-modules

$$i^{*}\mathscr{M}_c \xrightarrow{\sim} \mathbb{C}[W] \otimes i^{*}\mathscr{L}_c.$$

The composition $i^{*}\bigl(\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathscr{L}_c\bigr) \to i^{*}\mathscr{M}_c \xrightarrow{\sim} \mathbb{C}[W] \otimes i^{*}\mathscr{L}_c$ is given by $a \otimes u_c \mapsto \sum_{w\in W} w \otimes (w^{-1}a)u_c$ in virtue of (3.6). Then the lemma follows from the fact that $\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}] \to \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]$ given by $a \otimes b \mapsto \sum_{w\in W} w \otimes (w^{-1}a)b$ is an isomorphism. □

Lemma 3.3. *The morphism $\iota \colon \mathscr{L}_c \to \mathscr{M}_c$ is injective and its image is stable by eH_ce. Furthermore, eH_ce acts faithfully on $\mathscr{L}_c$.*

Proof. The injectivity of ι follows from Lemma 3.2, because $\mathscr{L}_c$ does not have a non-zero submodule supported in $(\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}) \times V$ by Lemma 3.1.

Since eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W$ and $\mathbf{y}^2e$ (cf. § 3.2.1), the stability result follows from the following result (cf. [4, Proposition 5.4.1] and [6, Proposition 6.2]):

$$\mathbf{y}^2v_c = \Delta_{\mathfrak{g}}v_c. \tag{3.7}$$

Here $\Delta_{\mathfrak{g}} = \displaystyle\sum_{i,j=1,\ldots,n} \frac{\partial^2}{\partial A_{ij}\partial A_{ji}}$ is the Laplacian on $\mathfrak{g}$.

Finally, the faithfulness of the action of eH_ce follows from the faithfulness of the action of H_c on $H_cv_c \subset \mathscr{M}_c$. With the notations of the proof of Lemma 3.2, we have an isomorphism $i^*\mathscr{M}_c \simeq \mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times V} \rtimes W$ compatible with the action of $\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}} \rtimes W$, and the faithfulness follows from that of θ_c. □

Remark 3.4. (i) In other words, the subalgebra of $\mathrm{End}_{\mathscr{D}_{\mathfrak{g}\times V}}(\mathscr{L}_c)$ generated by $\mathbb{C}[\mathfrak{t}]^W$ and by the endomorphism $u_c \mapsto \Delta_{\mathfrak{g}}u_c$ is isomorphic to eH_ce.

(ii) The action of eH_ce on $\mathscr{L}_c$ can be described as follows. Let $\kappa_0\colon \mathbb{C}[\mathfrak{t}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathscr{D}(\mathfrak{g})$ and $\kappa_1\colon \mathbb{C}[\mathfrak{t}^*]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathscr{D}(\mathfrak{g})$ be the canonical morphisms. We have

$$(ae)u_c = \kappa_0(a)u_c \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W, \qquad (b e)u_c = \kappa_1(b^*)u_c \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W. \tag{3.8}$$

The first equality is clear. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{t}]^W_k & \xrightarrow{\kappa_0} & \mathbb{C}[\mathfrak{g}]^G_k \\ {\scriptstyle(\mathrm{ad}(\mathbf{y}^2))^k}\downarrow & & \downarrow{\scriptstyle(\mathrm{ad}(\Delta_{\mathfrak{g}}))^k} \\ \mathbb{C}[\mathfrak{t}^*]^W_k & \xrightarrow[\kappa_1]{} & \mathbb{C}[\mathfrak{g}^*]^G_k. \end{array} \tag{3.9}$$

From (3.7) and the first equality, we deduce that

$$\big(\mathrm{ad}(\Delta_{\mathfrak{g}})\big)^k(\kappa_0(a))v_c = (-1)^k\big(\mathrm{ad}(\mathbf{y}^2)\big)^k(a)v_c$$

for $a \in \mathbb{C}[\mathfrak{t}]^W_k$. This gives the second equality.

3.3.2. The morphism ι gives rise to an $(H_c \otimes \mathscr{D}_{\mathfrak{g}\times V})$-linear morphism

$$H_ce \otimes_{eH_ce} \mathscr{L}_c \to \mathscr{M}_c. \tag{3.10}$$

Consider the conditions:

$$H_ceH_c = H_c, \tag{3.11}$$

$$eH_ce_{\mathrm{det}}H_ce = eH_ce \text{ and } e_{\mathrm{det}}H_ceH_ce_{\mathrm{det}} = e_{\mathrm{det}}H_ce_{\mathrm{det}}. \tag{3.12}$$

Lemma 3.5. *If* (3.11) *is satisfied, then the morphism* (3.10) *is injective.*

Proof. Since H_ce is a projective eH_ce-module, any coherent submodule of $H_ce \otimes_{eH_ce} \mathscr{L}_c$ vanishes as soon as it is zero on $\mathfrak{g}_{\mathrm{reg}} \times V$ by Lemma 3.1. Hence it is enough to show that the morphism (3.10) is injective on $\mathfrak{g}_{\mathrm{reg}} \times V$. Then the result follows from Lemma 3.2 and the fact that the multiplication map gives an isomorphism of right $(eH_ce \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W)$-modules

$$\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} eH_ce \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W \xrightarrow{\sim} H_ce \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_{\mathrm{reg}}]^W.$$

□

Proposition 3.6. *Condition* (3.11) *holds if and only if eH_c gives a Morita equivalence between H_c and eH_ce. Similarly, Condition* (3.12) *holds if and only if eH_ce_{det} gives a Morita equivalence between $e_{\mathrm{det}}H_ce_{\mathrm{det}}$ and eH_ce.*

This follows from the following Lemma:

Lemma 3.7. *Let A be a ring, and let e_1 and e_2 be idempotents in A. Assume that*

$$e_1Ae_2Ae_1 = e_1Ae_1 \quad \textit{and} \quad e_2Ae_1Ae_2 = e_2Ae_2.$$

(i) *For any A-module M, we have*

$$e_2Ae_1 \otimes_{e_1Ae_1} e_1M \xrightarrow{\sim} e_2M.$$

(ii) *e_1Ae_2 and e_2Ae_1 give a Morita equivalence between* $\mathrm{Mod}(e_1Ae_1)$ *and* $\mathrm{Mod}(e_2Ae_2)$.

Proof. (i) The surjectivity follows from $e_2M = e_2Ae_2M = e_2Ae_1Ae_2M \subset (e_2Ae_1)(e_1M)$.

Let us show its injectivity. By the assumption, there exists finitely many elements $a_i \in e_2Ae_1$ and $b_i \in e_1Ae_2$ such that $e_2 = \sum_i a_ib_i$. Consider now $u = \sum_j x_j \otimes v_j \in e_2Ae_1 \otimes_{e_1Ae_1} e_1M$ (where $x_j \in e_2Ae_1$, $v_j \in e_1M$). Assume $\sum_j x_jv_j = 0$. Then

$$u = \sum_{j,i} a_ib_ix_j \otimes v_j = \sum_{j,i} a_i \otimes b_ix_jv_j = 0.$$

(ii) It is enough to show that the multiplication maps $e_2Ae_1 \otimes_{e_1Ae_1} e_1Ae_2 \to e_2Ae_2$ and $e_1Ae_2 \otimes_{e_2Ae_2} e_2Ae_1 \to e_1Ae_1$ are isomorphisms. For the first one, we apply (i) to $M = Ae_2$. The second one can be handled similarly. $\square$

The previous result can be expressed in terms of bimodules:

Proposition 3.8. *Let A and B be rings, and let P be an (A,B)-bimodule, Q a (B,A)-bimodule and let $\varphi\colon P \otimes_B Q \to A$ be a morphism of (A,A)-bimodules, and $\psi\colon Q \otimes_A P \to B$ a morphism of (B,B)-bimodules. Assume that φ and ψ are surjective and that the following diagrams commute:*

$$\begin{tikzcd} P \otimes_B Q \otimes_A P \arrow[r, "\varphi\otimes P"] \arrow[d, "P\otimes\psi"] & A \otimes_A P \arrow[d, "\mathrm{can}"] \\ P \otimes_B B \arrow[r, "\mathrm{can}"'] & P \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Q \otimes_A P \otimes_B Q \arrow[r, "\psi\otimes Q"] \arrow[d, "Q\otimes\varphi"] & B \otimes_B Q \arrow[d, "\mathrm{can}"] \\ Q \otimes_A A \arrow[r, "\mathrm{can}"'] & Q. \end{tikzcd}$$

(i) *Then φ and ψ are isomorphisms, and P and Q give a Morita equivalence between* $\mathrm{Mod}(A)$ *and* $\mathrm{Mod}(B)$.

(ii) *Let M be an A-module and N a B-module, and let $f : Q \otimes_A M \to N$ and $g\colon P \otimes_B N \to M$ be morphisms such that the diagrams*

$$\begin{tikzcd} P \otimes_B Q \otimes_A M \arrow[r, "\varphi\otimes M"] \arrow[d, "P\otimes f"] & A \otimes_A M \arrow[d, "\mathrm{can}"] \\ P \otimes_B N \arrow[r, "g"] & M \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Q \otimes_A P \otimes_B N \arrow[r, "\psi\otimes N"] \arrow[d, "Q\otimes g"] & B \otimes_B N \arrow[d, "\mathrm{can}"] \\ Q \otimes_A M \arrow[r, "f"] & N. \end{tikzcd}$$

are commutative. Then f and g are isomorphisms.

Proof. Apply Lemma 3.7 to the ring $\begin{pmatrix} A & P \\ Q & B \end{pmatrix}$, its module $\begin{pmatrix} M \\ N \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $\square$

Remark 3.9. (i) It would be interesting to describe the image of the morphism (3.10).

(ii) Let

$$\mathscr{Y} = \{\frac{m}{d} \mid m, d \in \mathbb{Z}, 2 \leqslant d \leqslant n, (m,d) = 1, m < 0\}.$$

It is known that Condition (3.11) holds for $c \notin \mathscr{Y}$, while Condition (3.12) holds when $c-1 \notin \mathscr{Y}$, cf. [8, Theorem 3.3], [2, Theorem 8.1] and [3].

3.3.3. Let us consider the $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \otimes \mathscr{D}_{\mathfrak{g}\times V}$-linear morphism

$$\begin{aligned} \sigma\colon \mathscr{M}_c &\to \mathscr{M}_{c-1} \otimes \det(V) \\ v_c = q(A,z)^c \delta(x,A) &\mapsto q(A,z)\cdot q(A,z)^{c-1}\delta(x,A) \otimes l = q(A,z)v_{c-1} \otimes l. \end{aligned}$$

Here $l \in \det(V) := \bigwedge^n V$ is the element such that $q(A,z)l = A^{n-1}z \wedge A^{n-2}z \wedge \cdots \wedge Az \wedge z$. In particular, $q(A,z) \otimes l$ is a G-invariant section of $\mathscr{O}_{\mathfrak{g}\times V} \otimes \det(V)$.

So, the morphism σ is G-equivariant. We endow $\mathscr{M}_{c-1}$ with an H_c-module structure via the embedding $\theta_c\colon H_c \hookrightarrow \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$. Then σ is H_c-linear.

Remark 3.10. Note that $\mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ is an isomorphism on $\{q(A,z) \neq 0\}$. However, with our definition of $\mathscr{M}_c$, the morphism $\mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ is not a monomorphism for certain c, e.g. $c = 0$. Let us show this after restriction to $\mathfrak{t}_{\mathrm{reg}} \times V$. We have $q({}^tA, \partial_z)q(A,z)v_{c-1} = 0$ for $c = 0$ by (3.5), while the support of $q({}^tA, \partial_z)v_c$ is the subvariety $\{q(A,z) = 0\}$.

Let $\mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x)v_{c-1})$ be the $\mathscr{D}_{\mathfrak{g}\times V}$-submodule of $\mathscr{M}_{c-1}$ generated by $\mathfrak{d}(x)v_{c-1}$.

Lemma 3.11. (i) *$\mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x)v_{c-1})$ is invariant by $e_{\det}H_c e_{\det}$.*

(ii) *The morphism $\mathscr{L}_{c-1} \to \mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x)v_{c-1})$ given by $u_{c-1} \mapsto \mathfrak{d}(x)v_{c-1}$ is an isomorphism.*

Proof. Note that $e_{\det}\mathfrak{d}(x)v_{c-1} = \mathfrak{d}(x)v_{c-1}$. The proof is similar to that of Lemma 3.3: the key point is the following (cf. e.g. [13, Theorem 3.1])

$$\mathbf{y}^2(\mathfrak{d}(x)v_{c-1}) = \Delta_{\mathfrak{g}}(\mathfrak{d}(x)v_{c-1}). \tag{3.13}$$

□

By [2, Proposition 4.1], there is a (unique) isomorphism

$$f\colon e_{\det}H_c e_{\det} \xrightarrow{\sim} eH_{c-1}e$$

such that $\theta_{c-1}\big(f(a)\big) = \mathfrak{d}(x)^{-1}\theta_c(a)\mathfrak{d}(x)$ for $a \in e_{\det}H_c e_{\det}$.

The isomorphism $\mathscr{L}_{c-1} \xrightarrow{\sim} \mathscr{D}_{\mathfrak{g}\times V}(\mathfrak{d}(x)v_{c-1})$ of Lemma 3.11 is compatible with f and we will sometimes view $\mathscr{L}_{c-1}$ as an $(e_{\det}H_c e_{\det} \otimes \mathscr{D}_{\mathfrak{g}\times V})$-module.

By Lemma 3.2, the image of the morphism

$$e_{\det}H_c e \otimes_{eH_ce} \mathscr{L}_c|_{\mathfrak{g}_{\mathrm{reg}}\times V} \to \mathscr{M}_c|_{\mathfrak{g}_{\mathrm{reg}}\times V},\ a \otimes u_c \mapsto av_c$$

is contained in $\mathscr{D}_{\mathfrak{g}_{\mathrm{reg}}\times V}(\mathfrak{d}(x)v_c)$. It follows from Lemma 3.11 that over $\mathfrak{g}_{\mathrm{reg}} \times V$, the composite morphism $e_{\det}H_c e \otimes_{eH_ce} \mathscr{L}_c \to \mathscr{M}_c \to \mathscr{M}_{c-1} \otimes \det(V)$ factors through a morphism

$$\varphi\colon e_{\det}H_c e \otimes_{eH_ce} \mathscr{L}_c|_{\mathfrak{g}_{\mathrm{reg}}\times V} \longrightarrow \mathscr{L}_{c-1} \otimes \det(V)|_{\mathfrak{g}_{\mathrm{reg}}\times V}. \tag{3.14}$$

Similarly, we have the morphism

$$\begin{aligned} \psi\colon eH_c e_{\det} \otimes_{e_{\det}H_c e_{\det}} \mathscr{L}_{c-1} \otimes \det(V)|_{\{q(A,z)\neq 0\}} &\to \mathscr{L}_c|_{\{q(A,z)\neq 0\}} \\ a \otimes u_{c-1} \otimes l &\mapsto (a\mathfrak{d}(x))q(A,z)^{-1}u_c. \end{aligned} \tag{3.15}$$

The morphism φ is linear over $e_{\det}H_c e_{\det} \simeq eH_{c-1}e$ and the morphism ψ is linear over eH_ce. We have

$$\varphi(\mathfrak{d}(x)e \otimes u_c) = q(A,z)u_{c-1} \otimes l$$

and

$$q(A,z)\psi(\mathfrak{d}(x)e_{\det}\otimes u_{c-1}\otimes l)=\mathfrak{d}^2(A)u_c$$

where $\mathfrak{d}^2(A)$ is the discriminant of the characteristic polynomial of A.

Note that the following diagrams commute on $\mathfrak{g}_{\mathrm{reg}}\times V\cap\{q(A,z)\neq 0\}$:

$$\begin{tikzcd} eH_ce_{\det}\underset{e_{\det}H_ce_{\det}}{\otimes}e_{\det}H_ce\underset{eH_ce}{\otimes}\mathscr{L}_c \arrow[r] \arrow[d, "\varphi"] & eH_ce\underset{eH_ce}{\otimes}\mathscr{L}_c \arrow[d, "\mathrm{can}"] \\ eH_ce_{\det}\underset{e_{\det}H_ce_{\det}}{\otimes}\big(\mathscr{L}_{c-1}\otimes\det(V)\big) \arrow[r, "\psi"] & \mathscr{L}_c \end{tikzcd} \tag{3.16}$$

and

$$\begin{tikzcd} e_{\det}H_ce\underset{eH_ce}{\otimes}eH_ce_{\det}\underset{e_{\det}H_ce_{\det}}{\otimes}\big(\mathscr{L}_{c-1}\otimes\det(V)\big) \arrow[r] \arrow[d, "\psi"] & e_{\det}H_ce_{\det}\underset{e_{\det}H_ce_{\det}}{\otimes}\big(\mathscr{L}_{c-1}\otimes\det(V)\big) \arrow[d, "\mathrm{can}"] \\ e_{\det}H_ce\underset{eH_ce}{\otimes}\mathscr{L}_c \arrow[r, "\varphi"] & \mathscr{L}_{c-1}\otimes\det(V). \end{tikzcd} \tag{3.17}$$

Proposition 3.12. *The morphism φ extends uniquely to a morphism of $\mathscr{D}_{\mathfrak{g}\times V}$-modules:*

$$\varphi\colon e_{\det}H_ce\otimes_{eH_ce}\mathscr{L}_c\longrightarrow\mathscr{L}_{c-1}\otimes\det(V). \tag{3.18}$$

The proof will proceed by reduction to rank two. Recall that $\mathfrak{g}_1$ denotes the open subset of $\mathfrak{g}$ of matrices with at least $(n-1)$ distinct eigenvalues. Then $\mathfrak{g}\setminus\mathfrak{g}_1$ is a closed subset of $\mathfrak{g}$ of codimension 2.

We shall prove first the following lemma.

Lemma 3.13. *After restriction to $\mathfrak{g}_1\times V$, we have an inclusion of submodules of $\mathscr{M}_{c-1}$*

$$H_c\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c\subset\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c+\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1}$$

where $\overline{v}_c=q(A,z)v_{c-1}$.

Proof. Since $H_c=\mathbb{C}[\mathfrak{t}]\mathbb{C}[\mathfrak{t}^*]\mathbb{C}[W]$, it is enough to show that

$$\mathbb{C}[\mathfrak{t}^*]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c\in\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c+\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1}\quad\text{on }\mathfrak{g}_1\times V. \tag{3.19}$$

Here the action of $\mathbb{C}[\mathfrak{t}^*]$ is through $\mathbb{C}[\mathfrak{t}^*]\hookrightarrow H_c\xrightarrow{\theta_c}\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\rtimes W$.

Let us assume first that $n=2$. We have

$$q(A,z)=-A_{21}z_1^2+(A_{11}-A_{22})z_1z_2+A_{12}z_2^2.$$

We put

$$q(\partial_A,z)=-z_1^2\partial_{A_{12}}+z_1z_2(\partial_{A_{11}}-\partial_{A_{22}})+z_2^2\partial_{A_{21}}.$$

We will show that

$$(\partial_{x_1}-\partial_{x_2})q(A,z)v_{c-1}=-q(\partial_A,z)(x_1-x_2)v_{c-1}. \tag{3.20}$$

This is an equality in the $\mathscr{D}_{\mathfrak{g}\times V}$-submodule $\iota(\mathscr{L}_{c-1})$ of $\mathscr{M}_{c-1}$. Note that $(y_1-y_2)v_{c-1}=(\partial_{x_1}-\partial_{x_2})v_{c-1}$.

By § 3.2.3, we have

$$v_{c-1}=q(A,z)^{c-1}\delta(x_1+x_2-\mathrm{tr}(A))\delta(x_1x_2-\det(A)).$$

Since $q(\partial_A, z)q(A,z) = q(\partial_A, z)\operatorname{tr}(A) = 0$ and $q(\partial_A, z)\det(A) = -q(A,z)$, we obtain

$$q(\partial_A, z)v_{c-1} = q(A,z)^c\delta(x_1+x_2-\operatorname{tr}(A))\delta'(x_1x_2-\det(A)).$$

On the other hand, we have

$$(\partial_{x_1}-\partial_{x_2})q(A,z)v_{c-1} = (x_2-x_1)q(A,z)^c\delta(x_1+x_2-\operatorname{tr}(A))\delta'(x_1x_2-\det(A)).$$

The equality (3.20) then follows.

We assume now $n \geqslant 2$. Let S be the locally closed subset of $\mathfrak{g}$ of matrices

$$\begin{pmatrix} A' & 0 & 0 & \cdots & \\ 0 & a_3 & 0 & \cdots & \\ 0 & 0 & a_4 & & \\ \vdots & \vdots & & \ddots & \\ & & & & a_n \end{pmatrix}$$

where A' is a 2×2 matrix and $a_i \neq a_j$ $(3 \leqslant i < j \leqslant n)$ and a_i is not an eigenvalue of A' for $3 \leqslant i \leqslant n$. Let $\mathfrak{t}_1 = \mathfrak{t}\cap S = \{x \in \mathfrak{t}\,;\, x_i \neq x_j \text{ for } i<j \text{ and } 3 \leqslant j\}$. Let $x' = (x_1, x_2)$, $x'' = (x_3, \ldots, x_n)$ and $a'' = (a_3, \ldots, a_n)$.

We have $G\cdot S = \mathfrak{g}_1$. Let $i\colon S\times V \hookrightarrow \mathfrak{g}\times V$ be the inclusion map. Then, i is non-characteristic for $\mathscr{L}_c$ and $\mathscr{M}_{c-1}$, because we have $T_xS + T_x(G\cdot x) = T_x\mathfrak{g}$ for any $x \in S$.

Denote by $\mathfrak{g}'$ the subalgebra of $\mathfrak{g}$ of matrices (A_{ij}) with $A_{ij} = 0$ whenever $i > 2$ or $j > 2$. We identify $\mathfrak{g}'$ with $\mathfrak{gl}_2(\mathbb{C})$. Given an object $\mathcal{X}$ defined earlier for $\mathfrak{g}$, we denote by $\mathcal{X}'$ the corresponding objects for $\mathfrak{g}'$ (i.e., case $n = 2$). For example, W' is the subgroup of W generated by s_{12}.

Let $i''\colon \mathfrak{t}\times S \to \mathfrak{t}\times\mathfrak{g}$ be the embedding. We have an isomorphism of $\mathscr{D}_{\mathfrak{t}\times S}$-modules compatible with the action of W (cf. Proof of Lemma 3.2):

$$i''^*\mathscr{D}_{\mathfrak{t}\times\mathfrak{g}}\delta(x,A) \xrightarrow[\sim]{\delta(x,A)\mapsto\sum_w T_w^*\mathfrak{d}_1(A',a'')^{-1}\delta(x',A')\delta(x''-a'')} \bigoplus_{w\in W'\backslash W} T_w^*\mathscr{D}_{\mathfrak{t}\times S}\delta(x',A')\delta(x''-a'').$$

Here, T_w is the automorphism of $\mathfrak{t}$ given by w, and $\mathfrak{d}_1(A',a'') = \mathfrak{d}(a'')\prod_{i=3}^n\det(a_iI_2 - A')$, $\delta(x',A') = \delta(x_1+x_2-\operatorname{tr}(A'))\delta(x_1x_2-\det(A'))$.

Let $A \in S$. We have

$$q(A,z) = q'(A',z')\cdot q_1(A,z),$$

where

$$q_1(A,z) = (z_3\cdots z_n)\mathfrak{d}_1(A',a'').$$

Note that $\mathfrak{d}_1(A',a'')$ is invertible on S.

Let $p\colon \mathfrak{t}_{\mathrm{reg}}\times S\times V \to S\times V$ be the projection. We have a $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\otimes\mathscr{D}_{S\times V}$-linear isomorphism compatible with the action of W:

$$i^*\mathscr{M}_c \xrightarrow[\sim]{v_c\mapsto e\otimes\widetilde{v}_c} \mathbb{C}[W]\otimes_{\mathbb{C}[W']} p_*\big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c\big) \tag{3.21}$$

where $\widetilde{v}_c = v'_c\, q_1(A,z)^c\,\mathfrak{d}_1(A',a'')^{-1}\,\delta(x''-a'')$ with $v'_c = q'(A',z')^c\,\delta(x',A')$. Note that s_{12} acts trivially on $\widetilde{v}_c$. The action of $\mathscr{D}(\mathfrak{t}_{\mathrm{reg}})\rtimes W$ on $\mathbb{C}[W]\otimes_{\mathbb{C}[W']} p_*\big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c\big)$ is given by:

$$(a\otimes w)(w'\otimes s) = (ww')\otimes\big(((ww')^{-1}a)s\big) \quad \text{for } w,\, w' \in W,\, a \in \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}),\, s \in p_*\big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c\big).$$

Note that $\mathscr{D}_{S\times V}\widetilde{v}_c$ is stable by $\mathbb{C}[\mathfrak{t}_1]^{W'}$ as a submodule of $p_*\big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_c\big)$. Since $\mathbb{C}[\mathfrak{t}_1] = \mathbb{C}[\mathfrak{t}]\mathbb{C}[\mathfrak{t}_1]^{W'}$, $\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c$ is stable by $\mathbb{C}[\mathfrak{t}_1]$.

Let us still denote by $\widetilde{v}_c = q(A,z)\widetilde{v}_{c-1}$, the image of $\widetilde{v}_c$.

Let us set $\widetilde{y}_1 = \partial_{x_1} - c(x_1-x_2)^{-1}(1-s_{12})$ and $\widetilde{y}_2 = \partial_{x_2} - c(x_2-x_1)^{-1}(1-s_{12})$, partial Dunkl operators, and let R be the algebra generated by $\widetilde{y}_1$, $\widetilde{y}_2$ and ∂_{x_i} $(i=3,\dots,n)$. Then s_{12} acts on R by the permutation of $\widetilde{y}_1$ and $\widetilde{y}_2$. We have $R = R^{W'} \oplus (\widetilde{y}_1-\widetilde{y}_2)R^{W'}$.

Let

$$\begin{aligned}
\tilde{\mathscr{N}} &= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + (\widetilde{y}_1-\widetilde{y}_2)\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c \\
&= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(\widetilde{y}_1-\widetilde{y}_2)\widetilde{v}_c \\
&= \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(\partial_{x_1}-\partial_{x_2})\widetilde{v}_c
\end{aligned}$$

be a submodule of $p_*\Big(\mathscr{D}_{\mathfrak{t}_{\mathrm{reg}}\times S\times V}\widetilde{v}_{c-1}\Big)$. Since $(\widetilde{y}_1+\widetilde{y}_2)\widetilde{v}_c$, $\widetilde{y}_1\widetilde{y}_2\widetilde{v}_c$, and $\partial_{x_i}\widetilde{v}_c$ $(i=3,\dots,n)$ belong to $\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c$ (cf. Lemma 3.3), $\tilde{\mathscr{N}}$ is invariant by R.

Set $\mathscr{N} = \mathbb{C}[W]\otimes_{\mathbb{C}[W']}\tilde{\mathscr{N}}$. Let us show that $\mathscr{N}$ is invariant by the action of $\mathbb{C}[\mathfrak{t}^*] \subset H_c \subset \mathscr{D}(\mathfrak{t}_{\mathrm{reg}}) \rtimes W$. For any i, we have

$$y_i(w\otimes t) = w\otimes \partial_{x_{w^{-1}(i)}}t - c\sum_{k\neq i} w(1+s_{w^{-1}(i),w^{-1}(k)})\otimes (x_{w^{-1}(i)}-x_{w^{-1}(k)})^{-1}t$$

for any $w\in W$ and $t\in\tilde{\mathscr{N}}$. Since $(x_a-x_b)^{-1}\in\mathbb{C}[\mathfrak{t}_1]$ when a or b is in $\{3,\dots,n\}$, we have $y_i(w\otimes t)\in\mathscr{N}$ when $w^{-1}(i)\neq 1,2$. If $w^{-1}(i)=1$, then

$$\begin{aligned}
y_i(w\otimes t) &\equiv w\otimes\partial_{x_1}t - cw(1+s_{12})\otimes(x_1-x_2)^{-1}t \mod \mathscr{N} \\
&= w\otimes\widetilde{y}_1 t\in\mathscr{N}.
\end{aligned}$$

The case $w^{-1}(i)=2$ is similar. Hence we have shown that $\mathscr{N}$ is invariant by $\mathbb{C}[\mathfrak{t}^*]$. Thus, we obtain

$$\mathbb{C}[\mathfrak{t}^*](e\otimes\widetilde{v}_c)\subset\mathscr{N}.$$

The study of rank 2 above, i.e. (3.20), shows that

$$(\widetilde{y}_1-\widetilde{y}_2)\widetilde{v}_c\subset\mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}(x_1-x_2)\widetilde{v}_{c-1}.$$

Hence we obtain

$$\tilde{\mathscr{N}}\subset\tilde{\mathscr{N}}' := \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\widetilde{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{S\times V}\mathfrak{d}(x)\widetilde{v}_{c-1},$$

which implies

$$\mathbb{C}[\mathfrak{t}^*](e\otimes\widetilde{v}_c)\subset\mathscr{N}' := \mathbb{C}[W]\otimes\tilde{\mathscr{N}}'. \tag{3.22}$$

We have a commutative diagram, where the horizontal map is an isomorphism

$$\begin{tikzcd}
W\times_{W'}\mathfrak{t}_1 \arrow[rr, "(w,x)\mapsto(w(x),x)", "\sim"'] \arrow[dr, "(w,x)\mapsto w(x)"'] & & \mathfrak{t}\times_{\mathfrak{t}/W}\mathfrak{t}_1/W' \arrow[dl, "(x,x')\mapsto x"] \\
& \mathfrak{t} &
\end{tikzcd}$$

The diagram above is W-equivariant, for the action of $g\in W$ given by

$$g\cdot(w,x) = (gw,x) \text{ for } (w,x)\in W\times_{W'}\mathfrak{t}_1$$

$$g\cdot(x,x') = (g(x),x') \text{ for } (x,x')\in\mathfrak{t}\times_{\mathfrak{t}/W}\mathfrak{t}_1/W'.$$

It follows that we have an isomorphism of $\mathbb{C}[\mathfrak{t}]$-modules

$$\mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{t}_1]^{W'} \xrightarrow{\sim} \mathbb{C}[W] \otimes_{\mathbb{C}[W']} \mathbb{C}[\mathfrak{t}_1]$$
$$a \otimes a' \mapsto \sum_{w \in W/W'} w \otimes w^{-1}(a)a'.$$

In particular, we have $\mathbb{C}[W] \otimes_{\mathbb{C}[W']} \mathbb{C}[\mathfrak{t}_1] = \mathbb{C}[\mathfrak{t}] \cdot (e \otimes \mathbb{C}[\mathfrak{t}_1]^{W'})$. Since $\mathbb{C}[\mathfrak{t}_1]^{W'}\widetilde{v}_c \subset \mathscr{D}_{S\times V}\widetilde{v}_c$ and $\mathbb{C}[\mathfrak{t}_1]^{W'}\mathfrak{d}(x)\widetilde{v}_{c-1} \subset \mathscr{D}_{S\times V}\mathfrak{d}(x)\widetilde{v}_{c-1}$, we deduce that

$$\mathscr{N}' = \mathbb{C}[\mathfrak{t}]\Big(e \otimes \mathscr{D}_{S\times V}\widetilde{v}_c + e \otimes \mathscr{D}_{S\times V}\mathfrak{d}(x)\widetilde{v}_{c-1}\Big).$$

Together with (3.22), we obtain

$$\mathbb{C}[\mathfrak{t}^*]\mathscr{D}_{S\times V}(e \otimes \widetilde{v}_c) \subset \mathbb{C}[\mathfrak{t}]\Big(\mathscr{D}_{S\times V}(e \otimes \widetilde{v}_c) + \mathscr{D}_{S\times V}(e \otimes \mathfrak{d}(x)\widetilde{v}_{c-1})\Big).$$

Via the isomorphism (3.21), this shows that

$$i^*\Big(\mathbb{C}[\mathfrak{t}^*]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c\Big) \subset i^*\Big(\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c + \mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1}\Big).$$

Since $\mu^{-1}(0) \cap T^*_{S\times V}(\mathfrak{g}\times V) \subset T^*_{\mathfrak{g}\times V}(\mathfrak{g}\times V)$, the non-characteristic condition implies the desired result (3.19) (cf. § 3.1.2). □

Proof of Proposition 3.12. By Lemma 3.13, we have, on $\mathfrak{g}_1 \times V$,

$$\begin{aligned} e_{\det}H_c\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c &\subset e_{\det}\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c + e_{\det}\mathbb{C}[\mathfrak{t}]\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1} \\ &\subset \mathbb{C}[\mathfrak{t}]^W\mathfrak{d}(x)\mathscr{D}_{\mathfrak{g}\times V}\overline{v}_c + \mathbb{C}[\mathfrak{t}]^W\mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1} = \mathscr{D}_{\mathfrak{g}\times V}\mathfrak{d}(x)v_{c-1}, \end{aligned}$$

since $e_{\det}\mathbb{C}[\mathfrak{t}]e = \mathbb{C}[\mathfrak{t}]^W\mathfrak{d}(x)e$ and $e_{\det}\mathbb{C}[\mathfrak{t}]e_{\det} = \mathbb{C}[\mathfrak{t}]^W e_{\det}$. Hence φ extends to a morphism defined on $\mathfrak{g}_1 \times V$. Then the desired result follows from $\mathscr{H}^1_{(\mathfrak{g}\setminus\mathfrak{g}_1)\times V}(\mathscr{L}_{c-1}) = 0$ (Lemma 3.1). □

4. Cherednik algebras and Hilbert schemes

4.1. **Geometry of the Hilbert scheme.**

4.1.1. We refer to [23, 12] for basic results on Hilbert schemes of points on $\mathbb{C}^2$.

Let us recall that

$$\mathfrak{X} = \{(A, B, z, \zeta) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^*\,;\, \mathbb{C}\langle A, B\rangle z = V\}$$

is the set of stable points for the action of G on $T^*(\mathfrak{g}\times V)$, relative to the character det of G. The group G acts freely on $\mathfrak{X}$. Let $\mu_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{g}$ be the moment map:

$$\mu_{\mathfrak{X}}(A, B, z, \zeta) = -[A, B] - z \circ \zeta.$$

It is a smooth morphism. Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme classifying closed subschemes of $\mathbb{C}^2$ with length n. Then we have an isomorphism $\mathrm{Hilb}^n(\mathbb{C}^2) \xrightarrow{\sim} \mu_{\mathfrak{X}}^{-1}(0)/G$. Note that we have $\zeta = 0$ on $\mu_{\mathfrak{X}}^{-1}(0)$ (cf. [7, Lemma 2.3]).

We shall write Hilb instead of $\mathrm{Hilb}^n(\mathbb{C}^2)$ for short. Let us denote by $p \colon \mu_{\mathfrak{X}}^{-1}(0) \to \mathrm{Hilb}$ the quotient map.

Let us recall the construction of p. For $(A, B, z, \zeta) \in \mu_{\mathfrak{X}}^{-1}(0)$, we regard V as a $\mathbb{C}[X, Y]$-module by $X \mapsto A$ and $Y \mapsto B$. Then z gives an epimorphism $\mathbb{C}[X, Y] \twoheadrightarrow V$ of $\mathbb{C}[X, Y]$-modules. Hence V gives a closed subscheme of $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[X, Y])$ of length n, which is the corresponding point of Hilb.

Let $\pi\colon \mathrm{Hilb} \to (\mathfrak{t} \times \mathfrak{t}^*)/W$ be the Hilbert-Chow morphism. Then Hilb is a resolution of singularities of $(\mathfrak{t} \times \mathfrak{t}^*)/W \simeq (\mathbb{C}^2)^n/S_n$, the scheme of n unordered points in $\mathbb{C}^2$. We have canonical isomorphisms

$$\Gamma(\mu_{\mathfrak{X}}^{-1}(0), \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)})^G \xrightarrow{\sim} \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}}) \xrightarrow{\sim} \Gamma((\mathfrak{t} \times \mathfrak{t}^*)/W, \mathscr{O}_{(\mathfrak{t}\times\mathfrak{t}^*)/W}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W.$$

Let $(\mathfrak{t}\times\mathfrak{t}^*)_{\mathrm{reg}}$ be the open subset of $\mathfrak{t}\times\mathfrak{t}^*$ where the action of W is free. The Hilbert-Chow morphism π is an isomorphism over $(\mathfrak{t}\times\mathfrak{t}^*)_{\mathrm{reg}}/W$. Let $E{:=}\pi^{-1}\Big(\big((\mathfrak{t}\times\mathfrak{t}^*)\setminus(\mathfrak{t}\times\mathfrak{t}^*)_{\mathrm{reg}}\big)/W\Big)$ be the exceptional divisor. It is a closed irreducible hypersurface of Hilb. The line bundle L on Hilb associated with the G-equivariant line bundle $\mathscr{O}_{\mathfrak{X}}\otimes\det(V)$ on $\mathfrak{X}$ is a very ample line bundle on Hilb.

Let us set

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} = \big\{\phi(p) \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]\,;\, \phi(gp) = \det(g)\phi(p) \text{ for any } g \in G\big\}\,.$$

It is isomorphic to $\Gamma(\mathrm{Hilb}, L) \simeq (\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]\otimes\det(V))^G$. Let $i\colon \mathfrak{t}\times\mathfrak{t}^*\times V \hookrightarrow \mathfrak{g}\times\mathfrak{g}\times V\times V^*$ be the embedding with the last component $\zeta = 0$. Then $i^{-1}(\mu_{\mathfrak{X}}^{-1}(0))$ contains $(\mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}^* \cup \mathfrak{t} \times \mathfrak{t}^*_{\mathrm{reg}}) \times (\mathbb{C}^*)^n$. For any $\phi \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det}$, we have $(i^*\phi)(x, y, gz) = \det(g)(i^*\phi)(x, y, z)$ for any invertible diagonal matrix g. Hence we have

$$(i^*\phi)(x, y, z) = a(x, y)(z_1 \cdots z_n)$$

for some rational function $a(x, y)$ which is regular on $(\mathfrak{t}_{\mathrm{reg}}\times\mathfrak{t}^*)\cup(\mathfrak{t}\times\mathfrak{t}^*_{\mathrm{reg}})$, an open subset of $\mathfrak{t} \times \mathfrak{t}^*$ with complement of codimension 2. Hence we have

$$a(x, y) \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det} = \{a \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]\,;\, wa = \det(w)a \text{ for any } w \in W\}\,.$$

Thus we obtain a map which is known to be an isomorphism (cf. e.g. [7, Proposition 8.2.1]) and we denote its inverse by i_d:

$$\begin{aligned} \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} \otimes \det(V) &\xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det} \\ \phi \otimes l &\mapsto \langle l, z_1 \wedge \cdots \wedge z_n\rangle a. \end{aligned} \tag{4.1}$$

Similarly, we have an isomorphism (cf. e.g. [7, Lemma 2.7.3]) whose inverse we denote by i_s:

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W. \tag{4.2}$$

Summarizing, we have the following isomorphisms

$$\begin{array}{lclclcl} i_d & : & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det} & \xrightarrow{\sim} & \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} \otimes \det(V) & \simeq & \Gamma(\mathrm{Hilb}, L), \\ i_s & : & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W & \xrightarrow{\sim} & \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^G & \simeq & \mathscr{O}_{\mathrm{Hilb}}(\mathrm{Hilb}). \end{array} \tag{4.3}$$

4.1.2. For a subset Y of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ with cardinality n, set $p_Y = \det(x_k^i y_k^j)_{(i,j)\in Y,\, k=1,\dots n} \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}$ and $s_Y(A, B, z, \zeta) = \det(A^i B^j z)_{(i,j)\in Y} \in \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{G,\det} = L(\mathrm{Hilb})$. Then $\{p_Y\}_Y$ is a basis of $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}$ as a vector space and $i_d(p_Y) = s_Y$. The $\mathscr{O}_{\mathrm{Hilb}}$-module L is generated by $\{s_Y\}_Y$, where Y ranges over the set of Young diagrams of size n. Here we regard a Young diagram Y as a subset of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ such that $(i, j) \in Y$ as soon as $(i, j + 1)$ or $(i + 1, j)$ belongs to Y.

There is a canonical global section $\tau \in \Gamma(\mathrm{Hilb}; L^{\otimes -2})$ satisfying the following property:

$$i_d(a_1)i_d(a_2)\tau = i_s(a_1a_2) \quad \text{for any } a_1,\, a_2 \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}. \tag{4.4}$$

Note that τ is identified with a function on $\mu_{\mathfrak{X}}^{-1}(0)$ such that $\tau(gp) = \det(g)^{-2}\tau(p)$ ($p \in \mu_{\mathfrak{X}}^{-1}(0)$ and $g \in G$).

The exceptional divisor E coincides with the set of zeroes of τ, and we obtain an isomorphism

$$L^{\otimes 2} \xrightarrow{\sim} \mathscr{O}_{\mathrm{Hilb}}(-E).$$

Let us denote by $\mathfrak{d}^2(A)$ the discriminant of the characteristic polynomial of A, and similarly for $\mathfrak{d}^2(B)$. Then we have

$$i_d(\mathfrak{d}(x)) = q(A,z), \quad i_d(\mathfrak{d}(y)) = q(B,z), \quad i_s(\mathfrak{d}(x)^2) = \mathfrak{d}^2(A), \quad i_s(\mathfrak{d}(y)^2) = \mathfrak{d}^2(B).$$

Hence we have

$$\mathfrak{d}^2(A) = q(A,z)^2\tau \text{ and } \mathfrak{d}^2(B) = q(B,z)^2\tau.$$

Lemma 4.1. (i) *The hypersurface of $\mu_{\mathfrak{X}}^{-1}(0)$ defined by $q(A,z) = 0$ is irreducible, and $p^{-1}E \cap \{q(A,z) = 0\}$ is of codimension 2 in $\mu_{\mathfrak{X}}^{-1}(0)$.*

(ii) *The hypersurface of $\mu_{\mathfrak{X}}^{-1}(0)$ defined by $\mathfrak{d}^2(A) = 0$ is $p^{-1}E \cup \{q(A,z) = 0\}$.*

(iii) *$\mu_{\mathfrak{X}}^{-1}(0) \cap \{q(A,z) = q(B,z) = 0\}$ is of codimension 2 in $\mu_{\mathfrak{X}}^{-1}(0)$.*

Note that (i) follows from the fact that $q(A,z)$ does not vanish on the irreducible hypersurface $p^{-1}E$ of $\mu_{\mathfrak{X}}^{-1}(0)$, and $q(A,z)$ is irreducible on $\mu_{\mathfrak{X}}^{-1}(0) \setminus p^{-1}E$. Statement (iii) follows from [12, Lemma 3.6.2].

4.2. **W-algebras on the Hilbert scheme.**

4.2.1. In the preceding sections, we have regarded $\mathfrak{X}$, Hilb, etc. as schemes. Hereafter, we regard them as complex manifolds. Note that the previous constructions and results would remain valid in the analytic category. Let $\mathscr{W}_{\mathfrak{X}}$ be the $\mathscr{W}$-algebra on $\mathfrak{X}$ associated with $\mathscr{D}_{\mathfrak{g}\times V}$. Denoting by $\pi\colon \mathfrak{X} \to \mathfrak{g} \times V$ the projection, we have a ring homomorphism $\pi^{-1}\mathscr{D}_{\mathfrak{g}\times V} \to \mathscr{W}_{\mathfrak{X}}$ respecting the order filtration. The ring $\mathscr{W}_{\mathfrak{X}}$ is flat over $\pi^{-1}\mathscr{D}_{\mathfrak{g}\times V}$. The action of G on $\mathfrak{g}\times V$ induces an action of G on $\mathscr{W}_{\mathfrak{X}}$ and there is a quantized moment map $\mu_{\mathscr{W}}\colon \mathfrak{g} \to \mathscr{W}_{\mathfrak{X}}$.

We have morphisms

$$\kappa_0\colon \mathbb{C}[\mathfrak{t}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \to \mathscr{W}_{\mathfrak{X}}(\mathfrak{X})$$

and

$$\kappa_1\colon \mathbb{C}[\mathfrak{t}^*]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathscr{D}_{\mathfrak{g}}(\mathfrak{g}) \to \mathscr{W}_{\mathfrak{X}}(\mathfrak{X}).$$

Note that $\kappa_1(\mathbf{y}^2) = \Delta_{\mathfrak{g}}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, let $\mathbb{C}[\mathfrak{t}^*]^W_k$ be the homogeneous part of $\mathbb{C}[\mathfrak{t}^*]^W$ of degree k. Then κ_0 sends $\mathbb{C}[\mathfrak{t}]^W$ to $\mathscr{W}_{\mathfrak{X}}(0)$ and κ_1 sends $\mathbb{C}[\mathfrak{t}^*]^W_k$ to $\mathscr{W}_{\mathfrak{X}}(k)$ and we have the commutative diagrams:

$$\begin{tikzcd} \mathbb{C}[\mathfrak{t}]^W \arrow[r, "\kappa_0"] \arrow[dr] & \mathscr{W}_{\mathfrak{X}}(0) \arrow[d, "\sigma_0"] \\ & \mathscr{O}_{\mathfrak{X}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{C}[\mathfrak{t}^*]^W_k \arrow[r, "\kappa_1"] \arrow[dr, "\hbar^{-k}"'] & \mathscr{W}_{\mathfrak{X}}(k) \arrow[d, "\sigma_k"] \\ & \hbar^{-k}\mathscr{O}_{\mathfrak{X}} \end{tikzcd} \tag{4.5}$$

Let us consider $\mathscr{W}_{\mathfrak{X}} \otimes_{\mathscr{D}_{\mathfrak{g}\times V}} \mathscr{L}_c$, which we denote by the same letter $\mathscr{L}_c$. With the notation of § 2.4.2, we have $\mathscr{L}_c = \Phi_{c\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$. Hence $\mathscr{L}_c$ is a twisted G-equivariant $\mathscr{W}_{\mathfrak{X}}$-module with twist $c\,\mathrm{tr}$. Let u_c be the canonical section of $\mathscr{L}_c$, and set $\mathscr{L}_c(m) = \mathscr{W}_{\mathfrak{X}}(m)u_c$. Then we have an isomorphism

$$\mathscr{L}_c(0)/\mathscr{L}_c(-1) \xrightarrow{\sim} \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}.$$

The support of $\mathscr{L}_c$ is $\mu_{\mathfrak{X}}^{-1}(0)$. The $\mathscr{W}_{\mathfrak{X}}$-module $\mathscr{L}_c$ has a left action of eH_ce by Lemma 3.3. Via the anti-involution $h \mapsto h^*$ of H_c, we regard $\mathscr{L}_c$ as a $(\mathscr{W}_{\mathfrak{X}}, eH_ce)$-bimodule. Similarly, $\mathscr{L}_{c-1}$ has a structure of $(\mathscr{W}_{\mathfrak{X}}, e_{\det}H_ce_{\det})$-bimodule (Lemma 3.11). These actions are explicitly given by:

$$\begin{aligned} u_cea &= \kappa_0(a)u_c \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W \subset H_c, \\ u_ceb &= \kappa_1(b)u_c \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W \subset H_c. \end{aligned} \tag{4.6}$$

$$\begin{aligned} u_{c-1}e_{\det}a &= \kappa_0(a)u_{c-1} \quad \text{for } a \in \mathbb{C}[\mathfrak{t}]^W \subset H_c, \\ u_{c-1}e_{\det}b &= \kappa_1(b)u_{c-1} \quad \text{for } b \in \mathbb{C}[\mathfrak{t}^*]^W \subset H_c. \end{aligned} \tag{4.7}$$

Since $\mu_{\mathfrak{X}}^{-1}(0)$ is smooth, we have

$$\mathscr{E}xt^j_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c, \mathscr{W}_{\mathfrak{X}}) = 0 \text{ for } j \neq \operatorname{codim}_{\mathfrak{X}}(\mu_{\mathfrak{X}}^{-1}(0)).$$

Hence, for any closed subset $S \subset \mu_{\mathfrak{X}}^{-1}(0)$, we have by Lemma 2.1:

$$\mathscr{H}^j_S(\mathscr{L}_c) = 0 \text{ for } j < \operatorname{codim}_{\mu_{\mathfrak{X}}^{-1}(0)} S. \tag{4.8}$$

In (3.18) and (3.15), we defined the morphisms:

$$\varphi\colon \mathscr{L}_c \underset{eH_ce}{\otimes} eH_ce_{\det} \longrightarrow \mathscr{L}_{c-1} \otimes \det(V) \tag{4.9}$$

and

$$\psi\colon \big(\mathscr{L}_{c-1} \otimes \det(V)\big) \underset{e_{\det}H_ce_{\det}}{\otimes} e_{\det}H_ce\,|_{\{q(A,z)\neq 0\}} \longrightarrow \mathscr{L}_c\,|_{\{q(A,z)\neq 0\}}\,.$$

Proposition 4.2. *The morphism ψ extends uniquely to a morphism defined on $\mathfrak{X}$.*

Proof. We have

$$q(A,z)\psi(u_{c-1} \otimes a) = u_c \cdot (\mathfrak{d}(x)a)$$

for any $a \in e_{\det}H_ce$.

Now let us show that

$$(\operatorname{ad}(\Delta_{\mathfrak{g}})^k q(A,z))\psi(u_{c-1} \otimes a) = u_c \cdot ((\operatorname{ad}(\mathbf{y}^2)^k \mathfrak{d}(x))a) \tag{4.10}$$

holds on $\{q(A,z) \neq 0\}$ by the induction on k.

We have

$$\begin{aligned} &(\operatorname{ad}(\Delta_{\mathfrak{g}})^k q(A,z))\psi(u_{c-1} \otimes a) \\ &\quad = \Delta_{\mathfrak{g}}(\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\psi(u_{c-1} \otimes a) - (\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\Delta_{\mathfrak{g}}\psi(u_{c-1} \otimes a). \end{aligned}$$

The first term is calculated as

$$\begin{aligned} \Delta_{\mathfrak{g}}(\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\psi(u_{c-1} \otimes a) &= \Delta_{\mathfrak{g}} u_c \cdot ((\operatorname{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a) \\ &= u_c\mathbf{y}^2 \cdot ((\operatorname{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a) \\ &= u_c \cdot (\mathbf{y}^2(\operatorname{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))a). \end{aligned}$$

The second term is calculated as

$$\begin{aligned} (\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\Delta_{\mathfrak{g}}\psi(u_{c-1} \otimes a) &= (\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\psi(\Delta_{\mathfrak{g}} u_{c-1} \otimes a) \\ &= (\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\psi(u_{c-1}\mathbf{y}^2 \otimes a) \\ &= (\operatorname{ad}(\Delta_{\mathfrak{g}})^{k-1} q(A,z))\psi(u_{c-1} \otimes \mathbf{y}^2 a) \\ &= u_c \cdot ((\operatorname{ad}(\mathbf{y}^2)^{k-1}\mathfrak{d}(x))\mathbf{y}^2 a). \end{aligned}$$

Hence we obtain (4.10). In particular, letting k to be $n(n-1)/2$, the degree of $\mathfrak{d}(x)$, and using the fact that $\mathrm{ad}(\Delta_{\mathfrak{g}})^{n(n-1)/2}q(A,z)$ is equal to $q(\partial_A,z)$ up to a constant multiple (see e.g. (3.2) and the sentence below), we obtain

$$q(\partial_A,z)\psi(u_{c-1}\otimes a)=u_c\cdot(\mathfrak{d}(y)a). \tag{4.11}$$

Hence $\psi(u_{c-1}\otimes a)$ extends to a section of $\mathscr{L}_c$ outside $q(B,z)=0$.

Thus we have shown that $\psi(u_{c-1}\otimes a)$ is a section defined outside $\{q(A,z)=0\}\cap\{q(B,z)=0\}$. Since $\{q(A,z)=0\}\cap\{q(B,z)=0\}\cap\mu_{\mathfrak{X}}^{-1}(0)$ is of codimension 2 in $\mu_{\mathfrak{X}}^{-1}(0)$ (Lemma 4.1), it follows that $\psi(u_{c-1}\otimes a)$ extends to a global section of $\mathscr{L}_c$ by (4.8). □

Remark 4.3. (i) So, we have obtained a structure of $(e+e_{\det})H_c(e+e_{\det})$-module on $\mathscr{L}_c\oplus\mathscr{L}_{c-1}\otimes\det(V)$.

(ii) We have

$$\begin{aligned}
&\varphi(u_c\otimes e\mathfrak{d}(x))=q(A,z)u_{c-1},\\
&\varphi(u_c\otimes e\mathfrak{d}(y))=q(\partial_A,z)u_{c-1},\\
&q(A,z)\psi(u_{c-1}\otimes a)=u_c\cdot(\mathfrak{d}(x)a)\\
&q(\partial_A,z)\psi(u_{c-1}\otimes a)=u_c\cdot(\mathfrak{d}(y)a) \qquad \text{for } a\in e_{\det}H_ce.
\end{aligned} \tag{4.12}$$

(iii) The diagrams (3.16) and (3.17) commute on $\mathfrak{X}$.

By Propositions 3.12, 4.2 and Remark 4.3 (iii), we obtain the following proposition (see Proposition 3.8).

Proposition 4.4. *Assume Condition* (3.12) *holds. Then we have isomorphisms of twisted G-equivariant $\mathscr{W}_{\mathfrak{X}}$-modules with twist $c\,\mathrm{tr}$:*

$$\varphi\colon \mathscr{L}_c\otimes_{eH_ce}eH_ce_{\det}\xrightarrow{\sim}\mathscr{L}_{c-1}\otimes\det(V)$$

and

$$\psi\colon \big(\mathscr{L}_{c-1}\otimes\det(V)\big)\underset{e_{\det}H_ce_{\det}}{\otimes}e_{\det}H_ce\xrightarrow{\sim}\mathscr{L}_c.$$

4.2.2. Let us consider

$$\mathscr{A}_c=\big(p_*(\mathscr{E}nd_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c))^G\big)^{\mathrm{opp}}.$$

It is a W-algebra on Hilb by Proposition 2.8. Let $\mathscr{A}_c(0)$ be the subring of sections of order at most 0. For $m\in\mathbb{Z}$, $\mathscr{L}_{c+m}\otimes\det(V)^{\otimes -m}$ belongs to $\mathrm{Mod}^G_{c\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$ (cf. (2.4)). Set

$$\mathscr{A}_{c,c+m}=(p_*\mathscr{H}om_{\mathscr{W}_{\mathfrak{X}}}(\mathscr{L}_c,\mathscr{L}_{c+m}\otimes\det(V)^{\otimes -m}))^G.$$

Then $\mathscr{A}_{c,c+m}$ is an $(\mathscr{A}_c,\mathscr{A}_{c+m})$-bimodule. Let $\mathscr{A}_{c,c+m}(0)=(p_*\mathscr{H}om_{\mathscr{W}_{\mathfrak{X}}(0)}(\mathscr{L}_c(0),\mathscr{L}_{c+m}(0)\otimes\det(V)^{\otimes -m}))^G$. Then $\mathscr{A}_{c,c+m}(0)$ is an $\mathscr{A}_c(0)$-lattice of $\mathscr{A}_{c,c+m}$ and $\mathscr{A}_{c,c+m}(0)/\mathscr{A}_{c,c+m}(-1)\simeq L^{\otimes -m}$, the associated line bundle on Hilb to $\mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}\otimes\det(V)^{\otimes -m}$ (cf. Proposition 2.8 (iii)).

4.3. Affinity of $\mathscr{A}_c$.

4.3.1. As an application of Theorem 2.9, we obtain the following vanishing theorem.

Theorem 4.5. *Assume Condition* (3.12) *holds for $c+m$ (for all $m\in\mathbb{Z}_{>0}$).*

(i) *For any good $\mathscr{A}_c$-module $\mathscr{M}$, $\varprojlim_K H^i(K,\mathscr{M})=0$ for $i>0$. Here K ranges over compact subsets of* Hilb.

(ii) *Any good $\mathscr{A}_c$-module $\mathscr{M}$ is generated by global sections on any compact subset of* Hilb.

Proof. By Proposition 4.4, for any $m>0$, $\mathscr{L}_{c+m}$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m-1}\otimes\det(V)$ and $\mathscr{L}_{c+m-1}\otimes\det(V)$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m}$ in the category $\mathrm{Mod}^G_{(c+m)\,\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$. Hence $\mathscr{L}_{c+m}\otimes\det(V)^{\otimes -m}$ is a direct summand of a direct sum of copies of $\mathscr{L}_c$ and $\mathscr{L}_c$ is a direct summand of a direct sum of copies of $\mathscr{L}_{c+m}\otimes\det(V)^{\otimes -m}$ in the category $\mathrm{Mod}^G_{c\mathrm{tr}}(\mathscr{W}_{\mathfrak{X}})$ for any $m>0$. It follows that $\mathscr{A}_{c,c+m}$ is a direct summand of a direct sum of copies of $\mathscr{A}_c$ and $\mathscr{A}_c$ is a direct summand of a direct sum of copies of $\mathscr{A}_{c,c+m}$ for any $m>0$. Moreover $\mathscr{A}_{c,c+m}$ is a good $\mathscr{A}_c$-module whose symbol is $L^{\otimes -m}$.

Theorem 2.9 now gives the conclusion. □

4.3.2. Let us give an F-action on $\mathscr{W}_{\mathfrak{X}}$ by $\mathcal{F}_t(A_{ij})=tA_{ij}$, $\mathcal{F}_t(\partial_{A_{ij}})=t^{-1}\partial_{A_{ij}}$, $\mathcal{F}_t(z_i)=tz_i$, $\mathcal{F}_t(\partial_{z_i})=t^{-1}\partial_{z_i}$ and $\mathcal{F}_t(\hbar)=t^2\hbar$ for $t\in\mathbb{G}_{\mathrm{m}}=\mathbb{C}^\times$. Since $B_{ij}=\sigma_0(\hbar\partial_{A_{ji}})$ and $\zeta_i=\sigma_0(\hbar\partial_{z_i})$, the corresponding action of $\mathbb{G}_{\mathrm{m}}$ on $\mathfrak{X}$ is $T_t((A,B,z,\zeta))=(tA,tB,tz,t\zeta)$. Its induced $\mathbb{G}_{\mathrm{m}}$-action on Hilb coincides with the action induced by the scalar $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{C}^2$. We define the F-action on $\mathscr{L}_c$ by $\mathcal{F}_t(u_c)=u_c$.

Note that

$$\begin{aligned}\mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}])}(\mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}]) &\simeq \mathrm{End}_{\mathrm{Mod}_F(\mathscr{W}_{T^*(\mathfrak{g}\times V)}[\hbar^{1/2}])}(\mathscr{W}_{T^*(\mathfrak{g}\times V)}[\hbar^{1/2}])\\ &\simeq \mathbb{C}[\hbar^{-1/2}A_{ij},\hbar^{1/2}\partial_{A_{ij}},\hbar^{-1/2}z_i,\hbar^{1/2}\partial_{z_i}]\simeq\mathscr{D}(\mathfrak{g}\times V).\end{aligned}$$

The F-action on $\mathscr{W}_{\mathfrak{X}}$ is compatible with the G-action on $\mathscr{W}$, and hence $\mathscr{A}_c$ is also a W-algebra on Hilb with F-action (cf. Proposition 2.8 (iv)). We define the F-action on $\mathscr{L}_{c-1}\otimes\det(V)$ by $\mathcal{F}_t(u_{c-1}\otimes l)=t^{-n}u_{c-1}\otimes l$. Hence $\mathscr{A}_{c,c-1}$ has a structure of $\mathscr{A}_c$-module with F-action.

4.3.3. The $\big((e+e_{\det})H_c(e+e_{\det})\big)^{\mathrm{opp}}$-module structure on $\mathscr{L}_c\oplus\big(\mathscr{L}_{c-1}\otimes\det(V)\big)$ gives a ring homomorphism

$$(e+e_{\det})H_c(e+e_{\det})\xrightarrow{\alpha}\mathrm{End}_{\mathscr{A}_c}(\mathscr{A}_c\oplus\mathscr{A}_{c,c-1})^{\mathrm{opp}}.$$

Since it is not compatible with the F-action, we shall modify α.

Set

$$\widetilde{\mathscr{A}_c}=\mathscr{A}_c[\hbar^{1/2}]\quad\text{and}\quad\widetilde{\mathscr{A}_{c,c-1}}=\mathscr{A}_{c,c-1}[\hbar^{1/2}].$$

Let $H_c\xrightarrow{\beta}\mathbf{k}[\hbar^{1/2}]\otimes_{\mathbb{C}}H_c$ be the ring homomorphism given by $x_i\mapsto\hbar^{-1/2}\otimes x_i$, $y_i\mapsto\hbar^{1/2}\otimes y_i$, $w\mapsto 1\otimes w$ $(w\in W)$.

Lemma 4.6. *The composition*

$$\Phi\colon(e+e_{\det})H_c(e+e_{\det})\xrightarrow{\beta}\mathbf{k}[\hbar^{1/2}]\otimes_{\mathbb{C}}(e+e_{\det})H_c(e+e_{\det})\xrightarrow{\alpha}\mathrm{End}_{\widetilde{\mathscr{A}_c}}(\widetilde{\mathscr{A}_c}\oplus\widetilde{\mathscr{A}_{c,c-1}})^{\mathrm{opp}}$$

sends $(e+e_{\det})H_c(e+e_{\det})$ *to* $\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}\oplus\widetilde{\mathscr{A}_{c,c-1}})^{\mathrm{opp}}$.

Proof. First let us show that Φ sends eH_ce to $\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$. For a homogeneous element $a\in\mathbb{C}[\mathfrak{t}]^W$ of degree k, $\Phi(ae)(u_c)=\hbar^{-k/2}\tilde a(A)u_c$, where $\tilde a(A)$ is the element of $\mathbb{C}[\mathfrak{g}]^G$ such that $\tilde a|_{\mathfrak{t}}=a$. Since $\tilde a(A)$ is also homogeneous of degree k, $\hbar^{-k/2}\tilde a(A)$ is $\mathcal{F}$-invariant, and $\Phi(ae)$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. On the other hand, we have $\Phi(\mathbf{y}^2e)(u_c)=\hbar\Delta_{\mathfrak{g}}u_c$ and

$\hbar\Delta_{\mathfrak{g}}$ is $\mathcal{F}$-invariant. Hence $\Phi(\mathbf{y}^2 e)$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Since eH_ce is generated by $\mathbb{C}[\mathfrak{t}]^W e$ and $\mathbf{y}^2 e$, we have $\Phi(eH_ce) \subset \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})$.

Similarly, we have $\Phi(e_{\det} H_c e_{\det}) \subset \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_{c,c-1}})$.

Let us show that $\Phi(e\mathfrak{d}(x)) \in \mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}})$. This follows from $\Phi(e\mathfrak{d}(x))(u_c) = \hbar^{-n(n-1)/4} q(A,z) u_{c-1} \otimes l$, $\mathcal{F}_t(q(A,z)) = t^{n+n(n-1)/2} q(A,z)$ and $\mathcal{F}_t(u_{c-1} \otimes l) = t^{-n} u_{c-1} \otimes l$.

For $a \in e_{\det} H_c e$, let us show that $\Phi(a)\colon \widetilde{\mathscr{A}_{c,c-1}} \to \widetilde{\mathscr{A}_c}$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Since $\Phi(ae\mathfrak{d}(x))$ belongs to $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$, and since $\Phi(e\mathfrak{d}(x))|_{\{q(A,z)\neq 0\}}$ is an isomorphism in the category $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c}|_{\{q(A,z)\neq 0\}})$, it follows that $\Phi(a)|_{\{q(A,z)\neq 0\}}$ is in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c}|_{\{q(A,z)\neq 0\}})$. Hence we conclude that $\Phi(a)$ is in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$. Similarly, one shows that $\Phi(eH_ce_{\det})$ is contained in $\mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}})$. ☐

In particular we obtain a morphism of algebras

$$eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}.$$

We denote by $\tilde{\varphi}$ and $\tilde{\psi}$ the modified morphisms in $\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})$ given in Lemma 4.6:

$$\begin{aligned}\tilde{\varphi} \quad &: \quad \widetilde{\mathscr{L}_c} \otimes_{eH_ce} eH_ce_{\det} \longrightarrow \widetilde{\mathscr{L}_{c-1}} \otimes \det(V),\\ \tilde{\psi} \quad &: \quad \big(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)\big) \otimes_{e_{\det}H_ce_{\det}} e_{\det}H_ce \longrightarrow \widetilde{\mathscr{L}_c}.\end{aligned}$$

We define the order filtration $F(eH_ce)$ on eH_ce by assigning order $1/2$ to x_i and y_i. Then the morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is compatible with the order filtrations, and the symbol map $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W \simeq \mathrm{Gr}^F(eH_ce) \to \mathrm{Gr}^F \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}) \subset \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})[\hbar^{\pm 1/2}]$ coincides with $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W_k \xrightarrow{\hbar^{-k} i_s} \hbar^{-k}\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})$ by (4.5). Here $k \in \mathbb{Z}/2$.

Lemma 4.7. *The morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is an isomorphism.*

Proof. Note that the subspace $\mathrm{Gr}^F \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}) \subset \Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})[\hbar^{\pm 1/2}]$ is contained in $\oplus_{k\in\mathbb{Z}/2}\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k \hbar^{-k}$ where $\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k$ is the homogeneous part of weight $2k$ with respect to the $\mathbb{G}_{\mathrm{m}}$-action. Hence we have a chain of morphisms

$$\begin{aligned}&\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W \xrightarrow{\sim} \mathrm{Gr}^F(eH_ce)\\ &\qquad \to \mathrm{Gr}^F(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}) \hookrightarrow \oplus_{k\in\mathbb{Z}/2}\Gamma(\mathrm{Hilb}, \mathscr{O}_{\mathrm{Hilb}})_k \hbar^{-k} \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W.\end{aligned}$$

Since the composition is the identity, the map $\mathrm{Gr}^F(eH_ce) \to \mathrm{Gr}^F(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}})$ is bijective. Hence the morphism $eH_ce \to \mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})^{\mathrm{opp}}$ is an isomorphism. Note that $\bigcap_k F_k\big(\mathrm{End}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})\big) = 0$. ☐

Remark 4.8. A similar argument shows that there is an isomorphism

$$eH_ce_{\det} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_F(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \widetilde{\mathscr{A}_{c,c-1}}).$$

(See § 4.4.)

Let $\mathrm{o} \in (\mathfrak{t} \times \mathfrak{t}^*)/W$ be the image of the origin of $\mathfrak{t} \times \mathfrak{t}^*$. Then the Hilbert-Chow morphism $\pi\colon \mathrm{Hilb} \to (\mathfrak{t} \times \mathfrak{t}^*)/W$ is $\mathbb{C}^\times$-equivariant, and every point of $(\mathfrak{t} \times \mathfrak{t}^*)/W$ shrinks to o.

Now the following theorem is a consequence of Theorem 2.10.

Theorem 4.9. *Assume Condition* (3.12) *holds for* $c+m$*, for all* $m \in \mathbb{Z}_{>0}$ (*this will be the case if* $c \notin \frac{1}{n!}\mathbb{Z}_{<0}$). *We have quasi-inverse equivalences of categories between* $\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})$ *and* $\mathrm{Mod}_{\mathrm{coh}}(eH_ce)$

$$\begin{array}{rcl} \mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c}) & \xleftrightarrow{\sim} & \mathrm{Mod}_{\mathrm{coh}}(eH_ce) \\ \mathscr{M} & \mapsto & \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}) \\ \widetilde{\mathscr{A}_c} \otimes_{eH_ce} M & \leftarrow\!\shortmid & M. \end{array}$$

Under this equivalence, $\widetilde{\mathscr{A}_c}$ and $\widetilde{\mathscr{A}_{c,c-1}}$ correspond to eH_ce and $eH_ce_{\det}$, respectively.

Theorem 4.10. *Assume Condition* (3.12) *holds for* $c+m$ (*for all* $m \in \mathbb{Z}_{>0}$). *Assume also Condition* (3.11) *holds* (*these assumptions will be satisfied if* $c \notin \frac{1}{n!}\mathbb{Z}_{<0}$). *Let* $\mathcal{B}_c = \mathscr{E}nd_{\widetilde{\mathscr{A}_c}}(\widetilde{\mathscr{A}_c} \otimes_{eH_ce} eH_c)^{\mathrm{opp}}$. *We have quasi-inverse equivalences of categories between* $\mathrm{Mod}_F^{\mathrm{good}}(\mathcal{B}_c)$ *and* $\mathrm{Mod}_{\mathrm{coh}}(H_c)$

$$\begin{array}{rcl} \mathrm{Mod}_F^{\mathrm{good}}(\mathcal{B}_c) & \xleftrightarrow{\sim} & \mathrm{Mod}_{\mathrm{coh}}(H_c) \\ \mathscr{M} & \mapsto & \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\mathcal{B}_c)}(\mathcal{B}_c, \mathscr{M}) \\ \mathcal{B}_c \otimes_{H_c} M & \leftarrow\!\shortmid & M. \end{array}$$

Remark 4.11. It would be very interesting to have a more direct construction of $\widetilde{\mathscr{A}_c} \otimes_{eH_ce} eH_c$.

4.4. **W-algebras as fractions of** eH_ce**.** We explain how sections of $\widetilde{\mathscr{A}_c}$ over open subsets of Hilb can be obtained by inverting elements in the Cherednik algebra.

Let $\{F_j(H_c)\}_{j\in\mathbb{Z}/2}$ be the filtration of H_c consisting of elements of order $\leqslant j$, where we give order $1/2$ to x_i, y_i and order 0 to $w \in W$. Then we have a canonical isomorphism $\sigma \colon \mathrm{Gr}^F(H_c) \xrightarrow{\sim} \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*] \rtimes W$. We have induced filtrations on eH_ce and $eH_ce_{\det}$, and σ induces isomorphisms

$$\begin{array}{rcl} \mathrm{Gr}^F(eH_ce) & \xrightarrow{\sim} & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W, \\ \mathrm{Gr}^F(eH_ce_{\det}) & \xrightarrow{\sim} & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}. \end{array}$$

Composing with the morphism $\mathbb{C}[\mathfrak{t}\times\mathfrak{t}^*] \to \mathbb{C}[\mathfrak{t}\times\mathfrak{t}^*][\hbar^{-1/2}]$ given by $a(x,y) \mapsto a(\hbar^{-1/2}x, \hbar^{-1/2}y)$, we obtain homomorphisms

$$\begin{array}{rcl} \mathrm{Gr}^F(eH_ce) & \longrightarrow & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W[\hbar^{-1/2}], \\ \mathrm{Gr}^F(eH_ce_{\det}) & \longrightarrow & \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}[\hbar^{-1/2}]. \end{array}$$

We shall set $\widetilde{\mathscr{W}_{\mathfrak{X}}} = \mathscr{W}_{\mathfrak{X}}[\hbar^{1/2}]$ and $\widetilde{\mathscr{W}_{\mathfrak{X}}}(0) = \mathscr{W}_{\mathfrak{X}}(0) + \hbar^{1/2}\mathscr{W}_{\mathfrak{X}}(0)$. We set $\widetilde{\mathscr{L}_c} = \widetilde{\mathscr{W}_{\mathfrak{X}}} \otimes_{\mathscr{W}_{\mathfrak{X}}} \mathscr{L}_c$. Then $\widetilde{\mathscr{L}_c} \oplus \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$ has a structure of $(\widetilde{\mathscr{W}_{\mathfrak{X}}}, (e+e_{\det})H_c(e+e_{\det}))$-bimodule. The action of $eH_ce_{\det}$ is given by $\tilde{\varphi} \colon \widetilde{\mathscr{L}_c} \otimes_{eH_ce} eH_ce_{\det} \to \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$. On the other hand, we have canonical isomorphisms $\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \simeq \mathrm{Gr}^F(\widetilde{\mathscr{L}_{c-1}}) \xrightarrow{\sim} \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}]$. Here $F(\widetilde{\mathscr{L}_c})$ (resp. $F(\widetilde{\mathscr{L}_{c-1}})$) is the order filtration given by $F_k(\widetilde{\mathscr{L}_c}) = \hbar^{-k}\widetilde{\mathscr{W}_{\mathfrak{X}}}(0)u_c$ (resp. $F_k(\widetilde{\mathscr{L}_{c-1}}) = \hbar^{-k}\widetilde{\mathscr{W}_{\mathfrak{X}}}(0)u_{c-1}$) for $k \in \mathbb{Z}/2$.

We have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \otimes \mathrm{Gr}^F(eH_ce) \arrow[r] \arrow[d] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^W[\hbar^{-1/2}] \arrow[dd, "i_s"] \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c} \otimes eH_ce) \arrow[d] & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \arrow[r, "\sim"] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}].
\end{tikzcd} \tag{4.13}$$

The morphism $\tilde{\varphi}$ is order-preserving and we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \otimes \mathrm{Gr}^F(eH_ce_{\det}) \arrow[r] \arrow[d] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}[\hbar^{-1/2}] \arrow[dd, "i_d"] \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c} \otimes eH_ce_{\det}) \arrow[d, "\tilde{\varphi}"] & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)) \arrow[r, "\sim"] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \det(V).
\end{tikzcd} \tag{4.14}$$

Hence, for any $a \in eH_ce_{\det}$, the morphism $a \colon \widetilde{\mathscr{L}_c} \to \widetilde{\mathscr{L}_{c-1}} \otimes \det(V)$ is an isomorphism on $\{i_d(\sigma(a)) \neq 0\}$. Then, for $b \in eH_ce_{\det}$, we can define

$$ba^{-1} \in \mathrm{End}_{\mathrm{Mod}^G_{F,\,c\,\mathrm{tr}}(\widetilde{\mathscr{W}_X}|_{\{i_d(\sigma(a))\neq 0\}})}(\widetilde{\mathscr{L}_c}\,|_{\{i_d(\sigma(a))\neq 0\}})^{\mathrm{opp}}$$

as the composition

$$\begin{tikzcd}
\widetilde{\mathscr{L}_c} \arrow[dr, "b"] \arrow[dd, dotted, "ba^{-1}"'] & \\
& \widetilde{\mathscr{L}_{c-1}} \otimes \det(V). \\
\widetilde{\mathscr{L}_c} \arrow[ur, "a"'] &
\end{tikzcd}$$

Thus we obtain ba^{-1} as an F-invariant section of $\widetilde{\mathscr{A}_c}$ defined on $\{i_d(\sigma(a)) \neq 0\}$. Note that $ba^{-1} = bc(ac)^{-1}$ for a non-zero element $c \in e_{\det}H_ce$. Note also that the image of $ac \in eH_ce$ in $\Gamma(\mathrm{Hilb}; \widetilde{\mathscr{A}_c})$ is invertible only on $\{i_d(\sigma(a)) \neq 0\} \cap \{i_d(\sigma(c)) \neq 0\} \cap (\mathrm{Hilb} \setminus E)$.

Remark 4.12. The morphism $\tilde{\psi} \colon \big(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)\big) \otimes_{e_{\det}H_ce_{\det}} e_{\det}H_ce \to \widetilde{\mathscr{L}_c}$ is also order-preserving, and it induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}^F(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V)) \otimes \mathrm{Gr}^F(e_{\det}H_ce) \arrow[r] \arrow[d] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}] \otimes \det(V) \otimes \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}[\hbar^{-1/2}] \arrow[dd, "\tau\cdot i_d"] \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_{c-1}} \otimes \det(V) \otimes e_{\det}H_ce) \arrow[d, "\tilde{\psi}"] & \\
\mathrm{Gr}^F(\widetilde{\mathscr{L}_c}) \arrow[r, "\sim"] & \mathscr{O}_{\mu_{\mathfrak{X}}^{-1}(0)}[\hbar^{\pm 1/2}].
\end{tikzcd}$$

Hence, for any $b \in e_{\det}H_ce$, the morphism $b \colon \widetilde{\mathscr{L}_{c-1}} \otimes \det(V) \to \widetilde{\mathscr{L}_c}$ is never an isomorphism on the exceptional divisor E.

4.5. **Rank 2 case.** Let us consider the case $n = 2$. Let $x_0 = x_1 + x_2$, $x = x_1 - x_2$, $y_0 = (y_1 + y_2)/2$ and $y = (y_1 - y_2)/2 \in H_c$. Then $[y_0, x_0] = 1$, $[y, x] = 1 - 2cs$ where $s = s_{12}$. Since y, x and s commute with $\mathbb{C}[x_0, y_0]$, we have an isomorphism of algebras $\mathbb{C}[x_0, y_0] \otimes H'_c \xrightarrow{\sim} H_c$, where H'_c is the subalgebra of H_c generated by x, y and s.

We have

$$eH_ce_{\det}H_ce = eH_ce \iff H_ce_{\det}H_c = H_c \iff c \neq 1/2,$$
$$e_{\det}H_ceH_ce_{\det} = e_{\det}H_ce_{\det} \iff H_ceH_c = H_c \iff c \neq -1/2.$$

Indeed, the first equivalences follow from the fact that $ye_{\det}x - xe_{\det}y = e[y, x] = (1-2c)e$ and when $c = 1/2$, there is a one-dimensional representation with $x, y \mapsto 0$, $s \mapsto 1$. The second follows from the first by the isomorphism $H_c \simeq H_{-c}$ given by $s \mapsto -s$. It follows that Condition (3.12) is satisfied for all $c+n$ ($n \in \mathbb{Z}_{>0}$) if and only if $c \neq -1/2, -3/2, \dots$.

Note that x, $y \in \mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]^{W,\det}$ and $\mathrm{Hilb} = \{i_d(x) \neq 0\} \cup \{i_d(y) \neq 0\}$, because $\mu_{\mathfrak{X}}^{-1}(0) \cap \{q(A, z) = q(B, z) = 0\} \subset \{(A, B, z, 0) \in \mathfrak{X}\,;\, Az, Bz \in \mathbb{C}z\} = \emptyset$. Quantized symplectic coordinates of $\widetilde{\mathscr{A}_c}$ are given by

$$((ey)(ex)^{-1}, \hbar^{1/2}ex_0; -\hbar ex^2/2, \hbar^{1/2}ey_0) \quad \text{on } \{i_d(x) \neq 0\}$$

and

$$((ex)(ey)^{-1}, \hbar^{1/2}ex_0; \hbar ey^2/2, \hbar^{1/2}ey_0) \quad \text{on } \{i_d(y) \neq 0\}.$$

Indeed, we have $[-ex^2/2, (ey)(ex)^{-1}] = e$, because

$$(ey)(ex)^{-1}(ex^2) = (ey)(ex)^{-1}(ex)(e_{\det}x) = eyx \quad \text{and}$$

$$(ex^2)(ey)(ex)^{-1} = (ex^2y)(ex)^{-1} = (eyx^2 - 2ex)(ex)^{-1} = (eyx)(ex)(ex)^{-1} - 2e = eyx - 2e.$$

Note that this provides an isomorphism $\mathrm{Hilb} \xrightarrow{\sim} T^*(\mathbb{P}^1 \times \mathbb{C})$. The projection $\mathrm{Hilb} \to \mathbb{P}^1$ is given by $[i_d(x) : i_d(y)]$ with the notation of homogeneous coordinates. By the isomorphism above, we have $E \simeq T^*_{\mathbb{P}^1}\mathbb{P}^1 \times T^*\mathbb{C}$.

Note that $(xe)^{-1}(ye)$ is invertible only on $\{i_s(x^2) \neq 0\} = \{i_d(x) \neq 0\} \setminus E$ for $c \neq -1/2$, because $exyx = ex(xy + 1 - 2cs) = ex^2y + (1 + 2c)ex$ and $(xe)^{-1}(ye) = (x^2e)^{-1}(xye) = (ex^2)^{-1}(exyx)(ex)^{-1} = (ey)(ex)^{-1} + (1 + 2c)(ex^2)^{-1}$.

Set $(a, \partial_a) = ((ey)(ex)^{-1}, -ex^2/2)$ and $(b, \partial_b) = (((ex)(ey)^{-1}, ey^2/2)$ and $\lambda = c - 1/2$. Then we have

$$b = a^{-1} \quad \text{and} \quad \partial_b = -a(a\partial_a - \lambda). \tag{4.15}$$

Indeed, we have

$$\begin{aligned} -a(a\partial_a - \lambda) &= (ey)(ex)^{-1}\big((ey)(ex)^{-1}(ex^2)/2 + c - 1/2\big) \\ &= (1/2)(ey)(ex)^{-1}(eyx + 2c - 1) = (1/2)(ey)(ex)^{-1}(exy) = ey^2/2. \end{aligned}$$

Recall that $\mathrm{o} \in (\mathfrak{t} \times \mathfrak{t}^*)/W$ is the image of the origin of $\mathfrak{t} \times \mathfrak{t}^*$. The inverse image $\pi^{-1}(\mathrm{o})$ by the Hilbert-Chow morphism π is $T^*_{\mathbb{P}^1}\mathbb{P}^1 \times \{0\} \subset T^*\mathbb{P}^1 \times T^*\mathbb{C}$. We identify it with $\mathbb{P}^1$. Then, (4.15) gives an isomorphism

$$\mathscr{E}nd_F(\widetilde{\mathscr{A}_c})\,|_{\pi^{-1}(\mathrm{o})} \xrightarrow{\sim} \mathscr{D}_{\mathbb{P}^1,\lambda} \otimes \mathbb{C}[x_0, y_0]$$

with $\lambda = c - 1/2$. Here, $\mathscr{D}_{\mathbb{P}^1,\lambda}$ is the twisted ring of differential operators (e.g. see [17, § 2]). If λ is an integer, then $\mathscr{D}_{\mathbb{P}^1,\lambda} \simeq \mathscr{O}_{\mathbb{P}^1}(\lambda) \otimes \mathscr{D}_{\mathbb{P}^1} \otimes \mathscr{O}_{\mathbb{P}^1}(-\lambda)$. Hence, we have a ring isomorphism $eH'_ce \simeq \Gamma(\mathbb{P}^1; \mathscr{D}_{\mathbb{P}^1,\lambda})$ and an equivalence $\mathrm{Mod}^{\mathrm{good}}_F(\widetilde{\mathscr{A}_c}) \simeq \mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_{\mathbb{P}^1,\lambda} \otimes \mathbb{C}[x_0, y_0])$. It is

well-known (cf. e.g. [17, § 7]) that $\mathrm{Mod}_{\mathrm{good}}(\mathscr{D}_{\mathbb{P}^1,\lambda})$ is equivalent to $\mathrm{Mod}_{\mathrm{coh}}(\Gamma(\mathbb{P}^1;\mathscr{D}_{\mathbb{P}^1,\lambda}))$ if and only if $\lambda \neq -1,-2,\ldots$ (i.e. $c \neq -1/2,-3/2,\ldots$).

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